

Regimes of steady-state turbulence in a quantum gas

Ashton Bradley

University of Padova, 20 July 2026



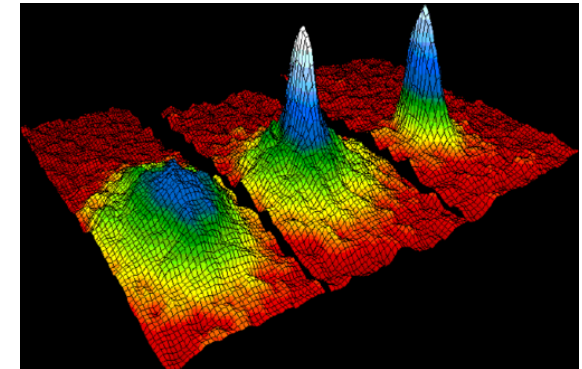
TE WHAI AO
DODD-WALLS CENTRE
for Photonic and Quantum Technologies



University
of Otago
ŌTĀKOU WHAKAIHU WAKA

University of Otago, Ōtākou Whaiku Waka

- **1869** First university in New Zealand. Now around 20,000 students, in a city of 120,000.
- **1998** First BEC in Sth. hemisphere
- Strength in quantum experiments and theory
- Host institute for national centre of excellence linking with Universities in Auckland, Waikato, Wellington, Christchurch



TE WHAI AO
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for Photonic and Quantum Technologies

A night photograph of a large, historic university building with a prominent clock tower, illuminated by warm lights. The building is surrounded by lush green trees and a well-maintained lawn. In the background, a modern multi-story building is visible under a dark sky with some clouds. The overall scene conveys a sense of academic excellence and research.

Physics Research

Antarctic, Climate, Space

Space Plasma

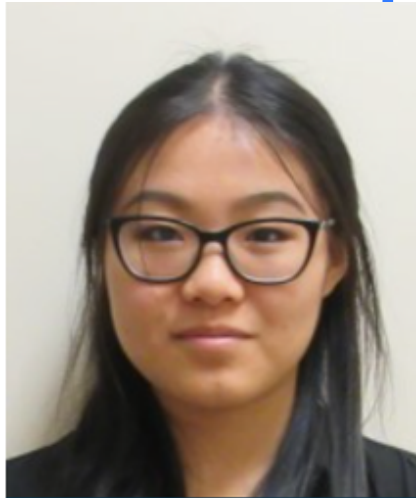
Instrumentation, Inverse problems

Sustainable Energy

Atomic and Optical Physics

Quantum and Condensed Matter Theory

Quantum Fluids Group



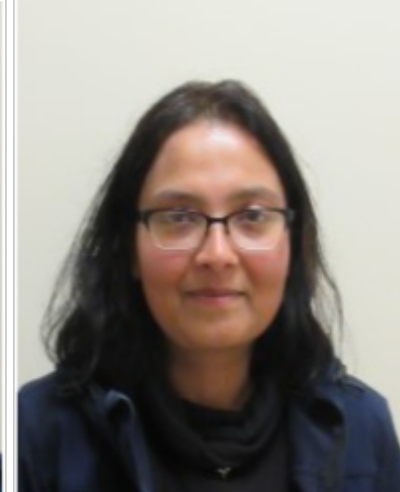
Cassie Dong
Honours Student



Tommy Fischer
MSc Student



Nils Krause
PhD Student



Dr Sukla Pal
Postdoctoral Fellow

Motivation

Regimes of quantum fluid turbulence

History



“La Turbolenza”

Da Vinci, 1508-1513, *Codex Leicester*

Richardson cascade

“Big whorls have little whorls that feed on their velocity, And little whorls have lesser whorls, And so on to viscosity.”

(Richardson 1922)

Kolmogorov 1941

Energy spectrum in steady turbulence has $k^{-5/3}$ power law

Turbulence spectra from a tidal channel

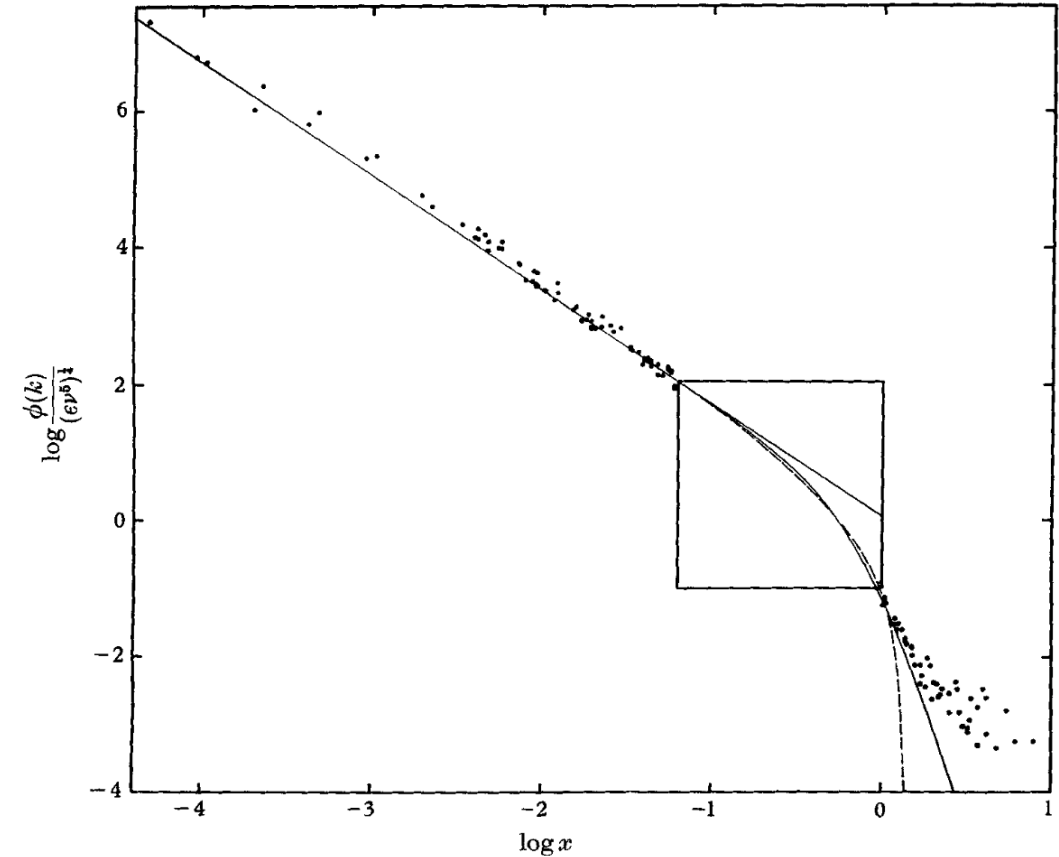
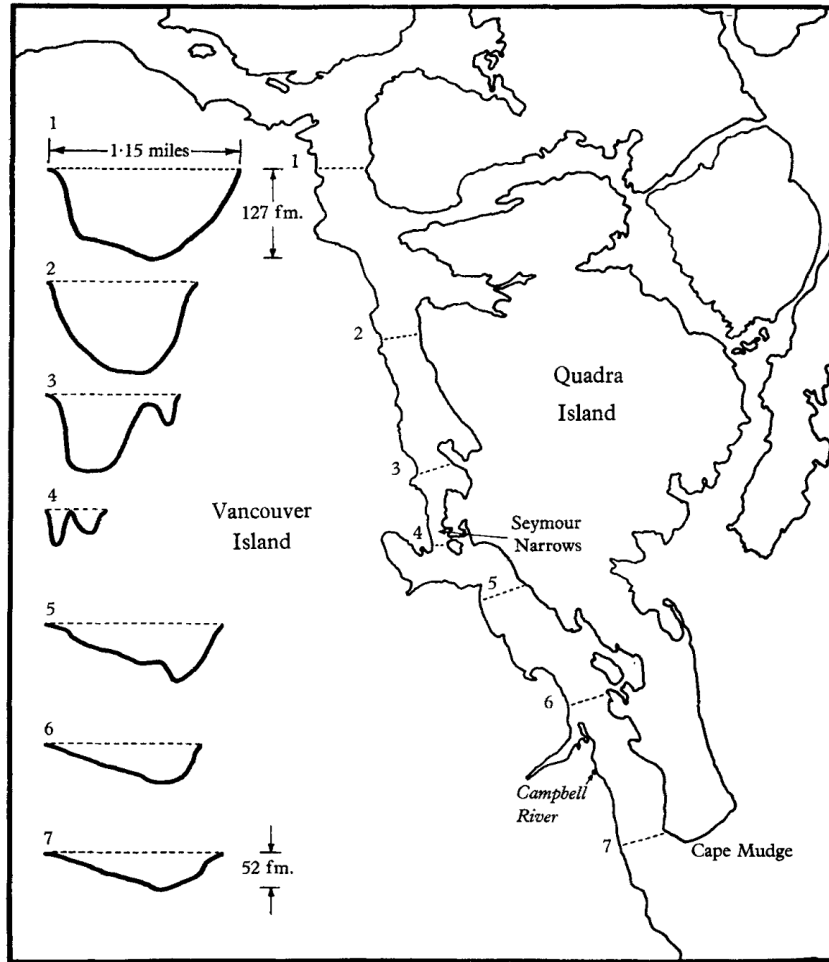


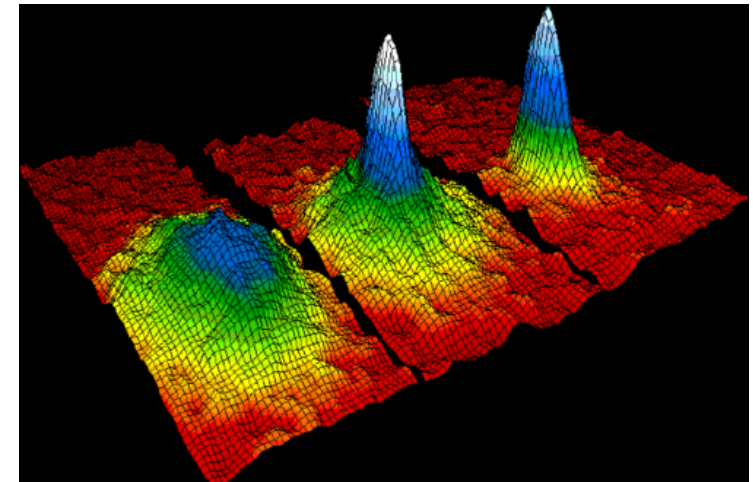
FIGURE 12. Seventeen spectra compared to the theories of Kolmogoroff, Heisenberg and Kovasznay. The straight line has a slope of $-\frac{5}{3}$, the curved solid line is Heisenberg's theory and the dashed line is Kovasznay's theory. Within the square, the observations are too crowded to display on this scale and they are shown in figure 13.

Quantum fluids are different ...

Classical Fluid

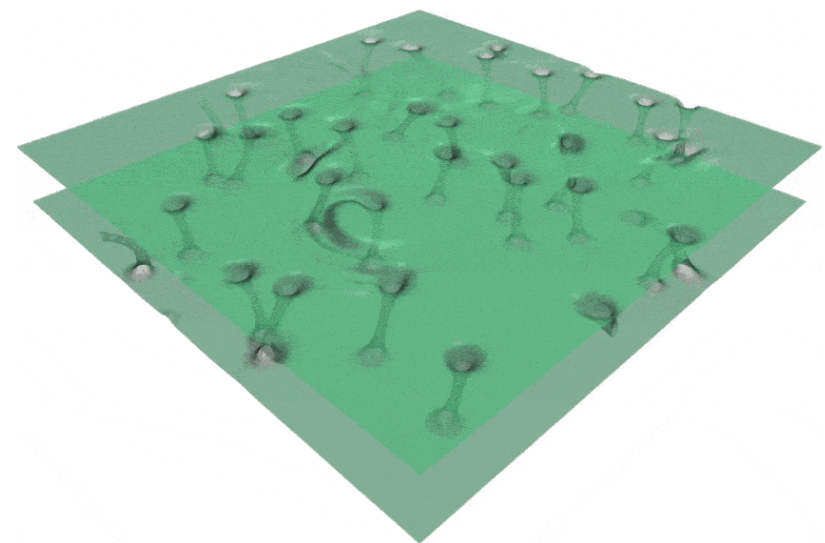
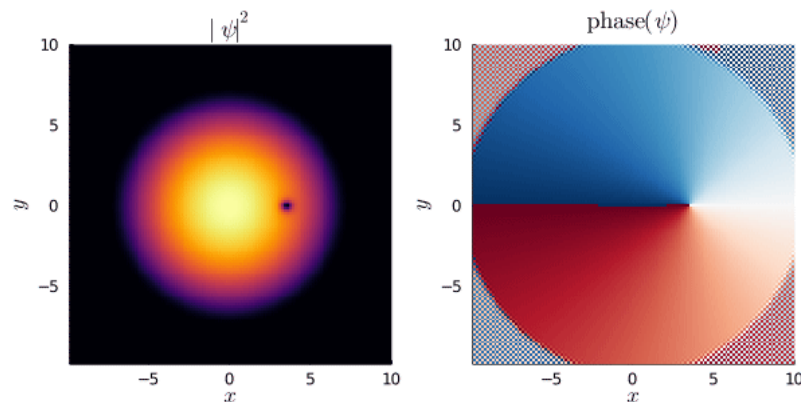
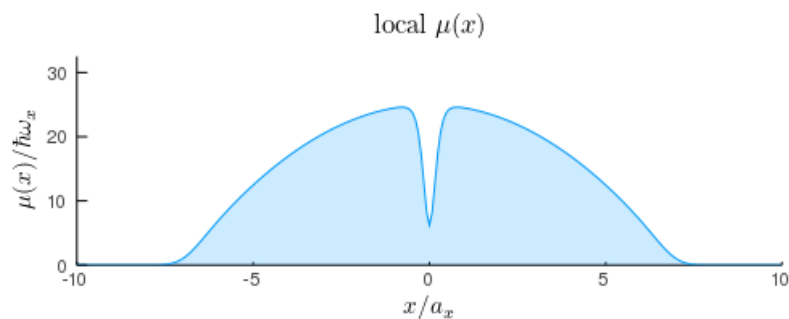


Quantum Fluid



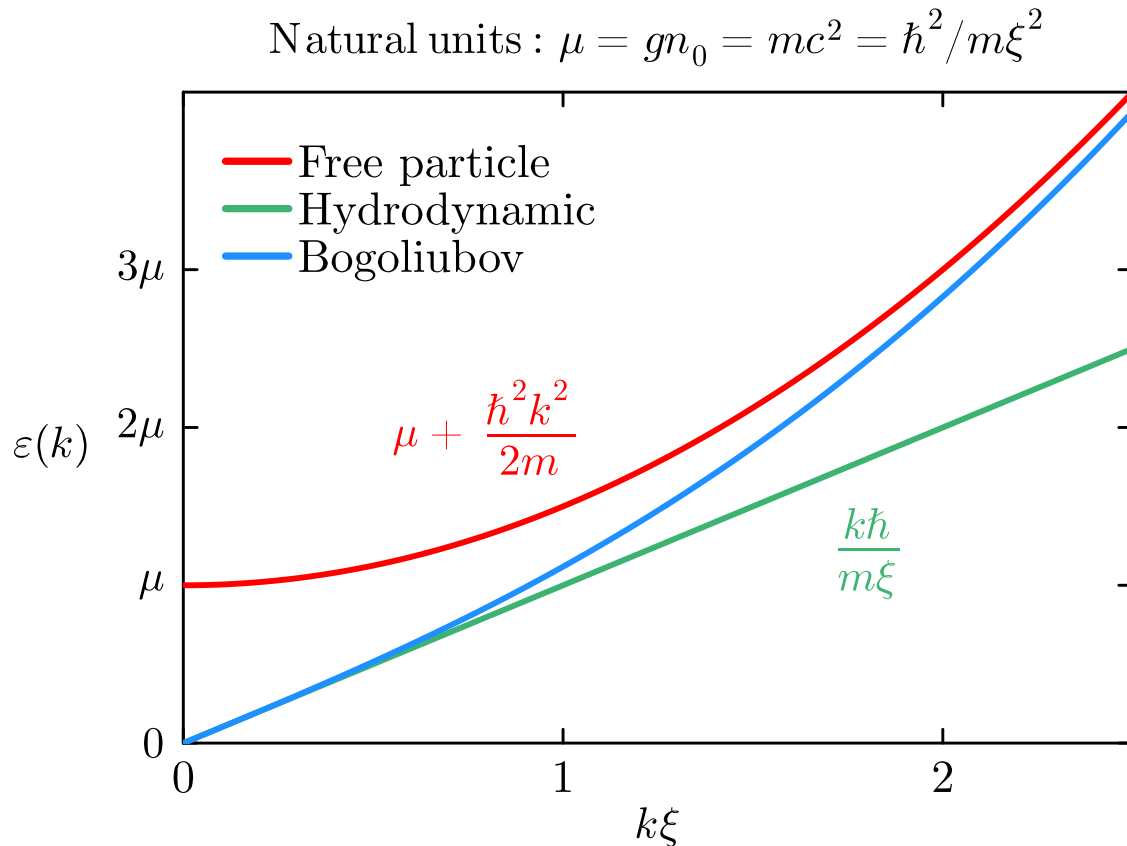
Gross-Pitaevskii Equation (generalised)

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2 \nabla^2}{2m} \psi + V\psi + g|\psi|^2 \psi$$

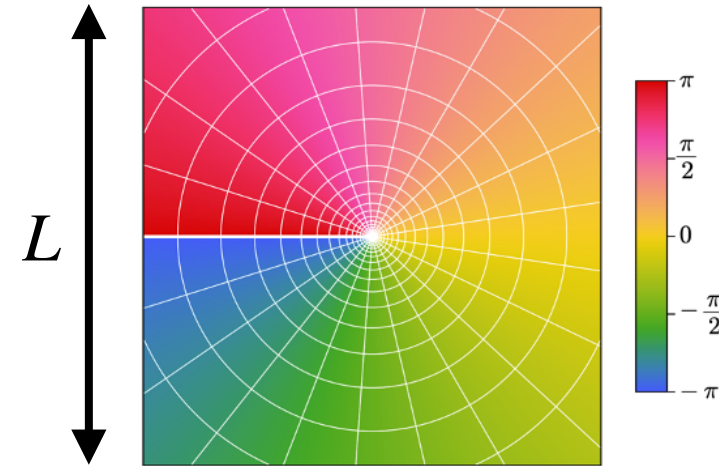


BEC excitations

Phonons



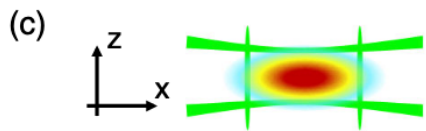
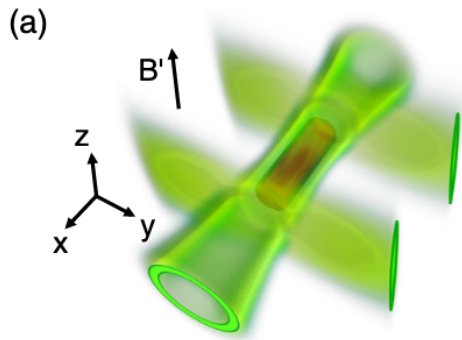
Quantum vortices



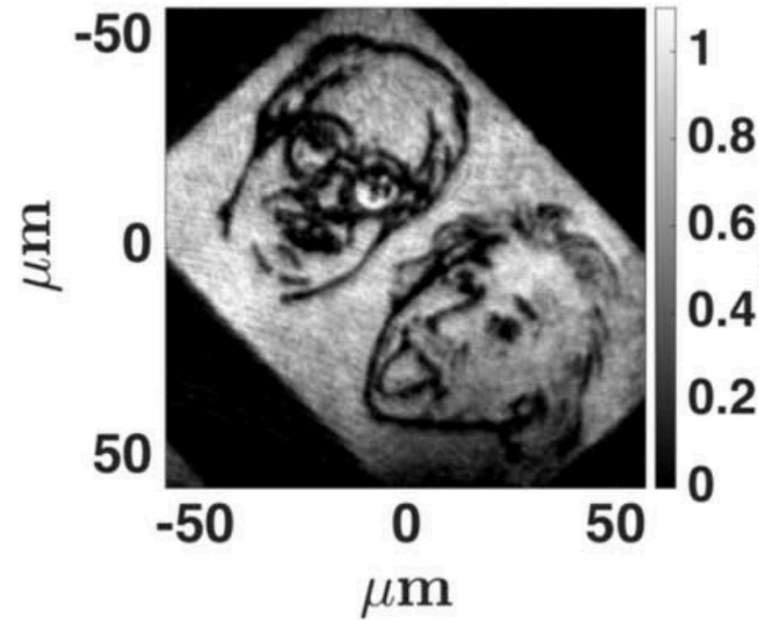
- Wavefunction $\psi = \sqrt{n}e^{i\Theta} \sim e^{i\theta}$
- Global flow $\mathbf{v} = \frac{\hbar}{m} \nabla \Theta = \frac{\hbar}{mr} \hat{\theta}$
- Energy $E \sim \log(L)$
- Healing length ξ , where $\mu = gn = \frac{\hbar^2}{m\xi^2}$

Box traps access new regimes

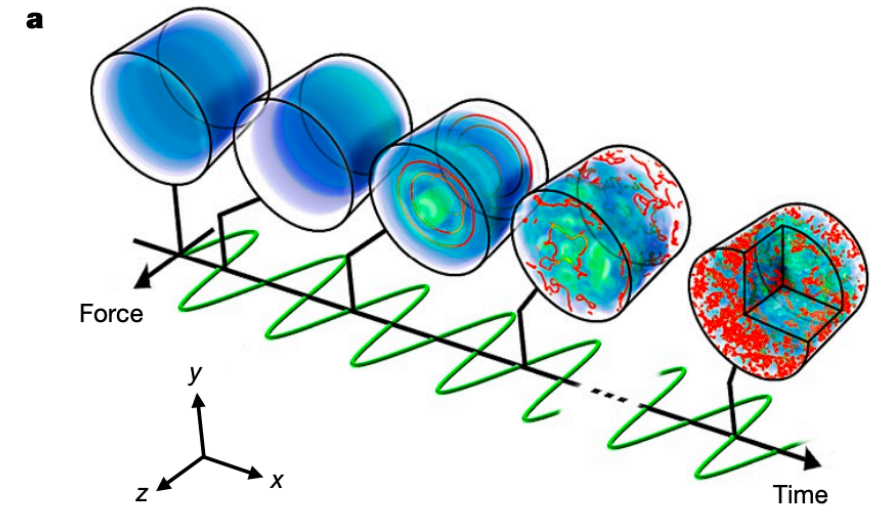
- **Quantum vortices** are vulnerable to dissipation
- **Box traps** remove/weaken these mechanisms
- **Novel phases** of quantum vortices become accessible



Gaunt et al, PRL (2013)



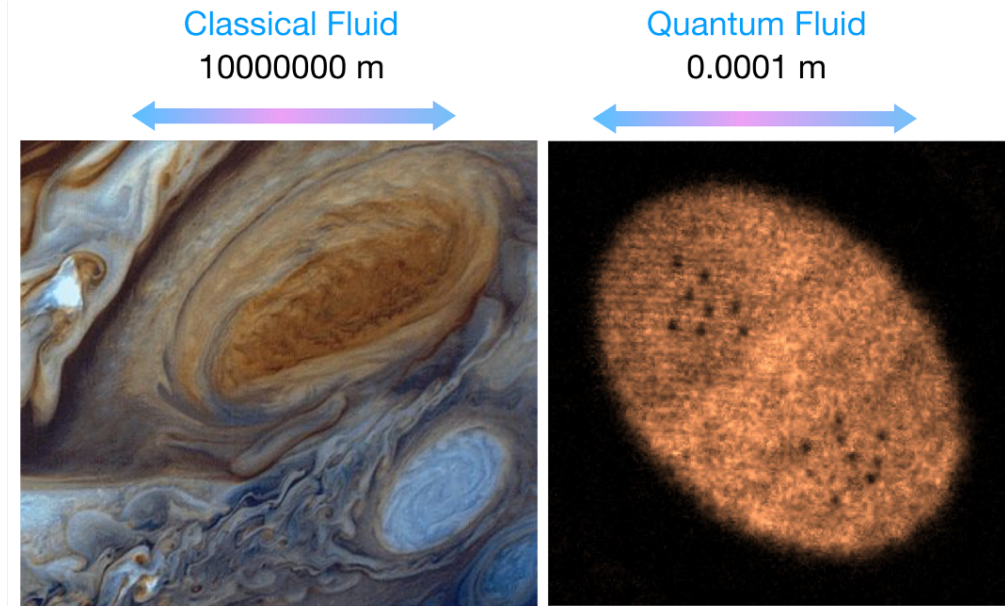
Gauthier et al, Optica (2016)



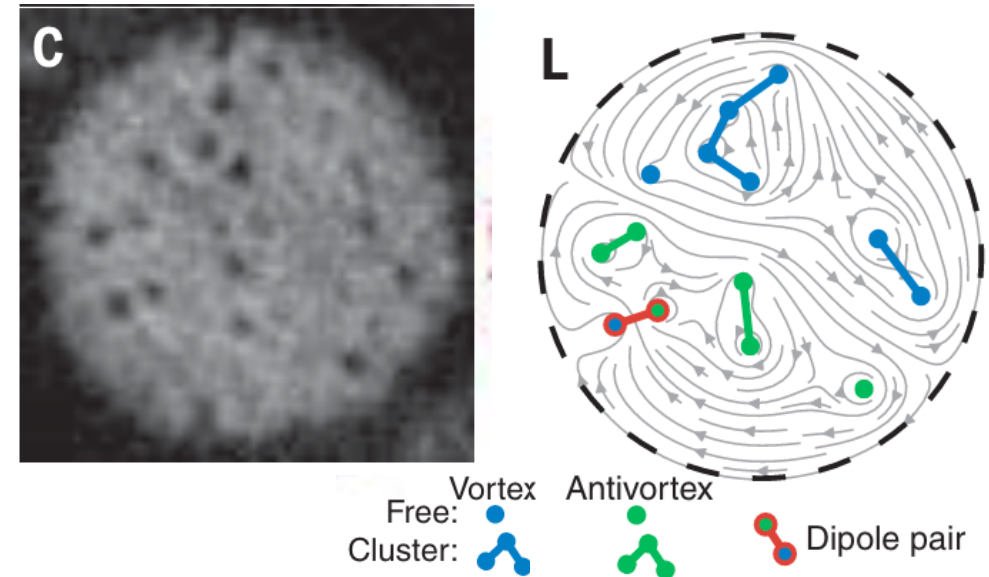
Navon et al, Nature (2016)

Box trap: Onsager Vortex Clusters

Statistical Mechanics of Point Vortices (Onsager)



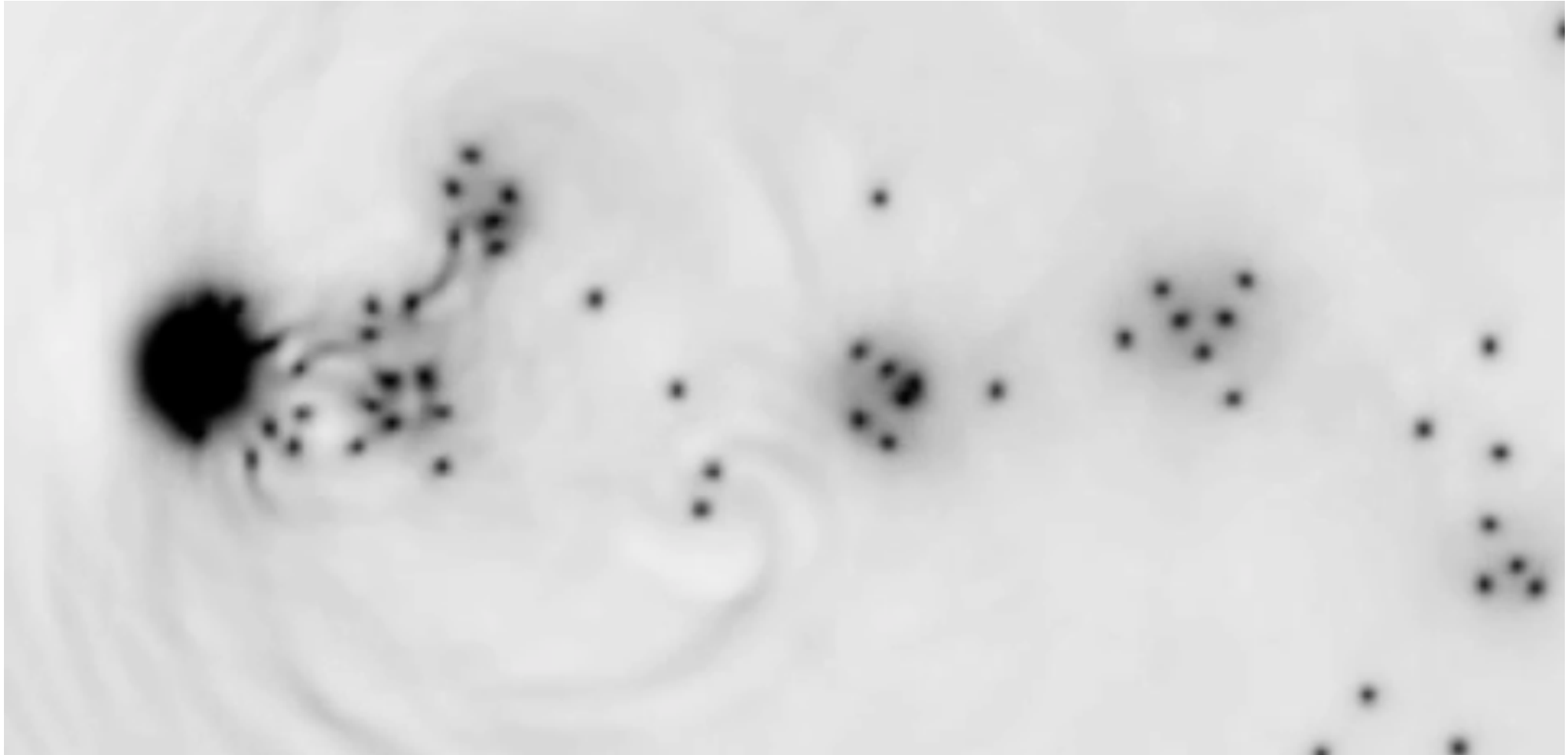
G. Gauthier, M. T. Reeves, X. Yu, A. S. Bradley, M. A. Baker, T. A. Bell, H. Rubinsztein-Dunlop, M. J. Davis, and T. W. Neely, Giant Vortex Clusters in a Two-Dimensional Quantum Fluid, Science 364, 1264 (2019).



S. P. Johnstone, A. J. Groszek, P. T. Starkey, C. J. Billington, T. P. Simula, and K. Helmerson, Evolution of Large-Scale Flow from Turbulence in a Two-Dimensional Superfluid, Science 364, 1267 (2019).

Forced 2D Quantum Turbulence

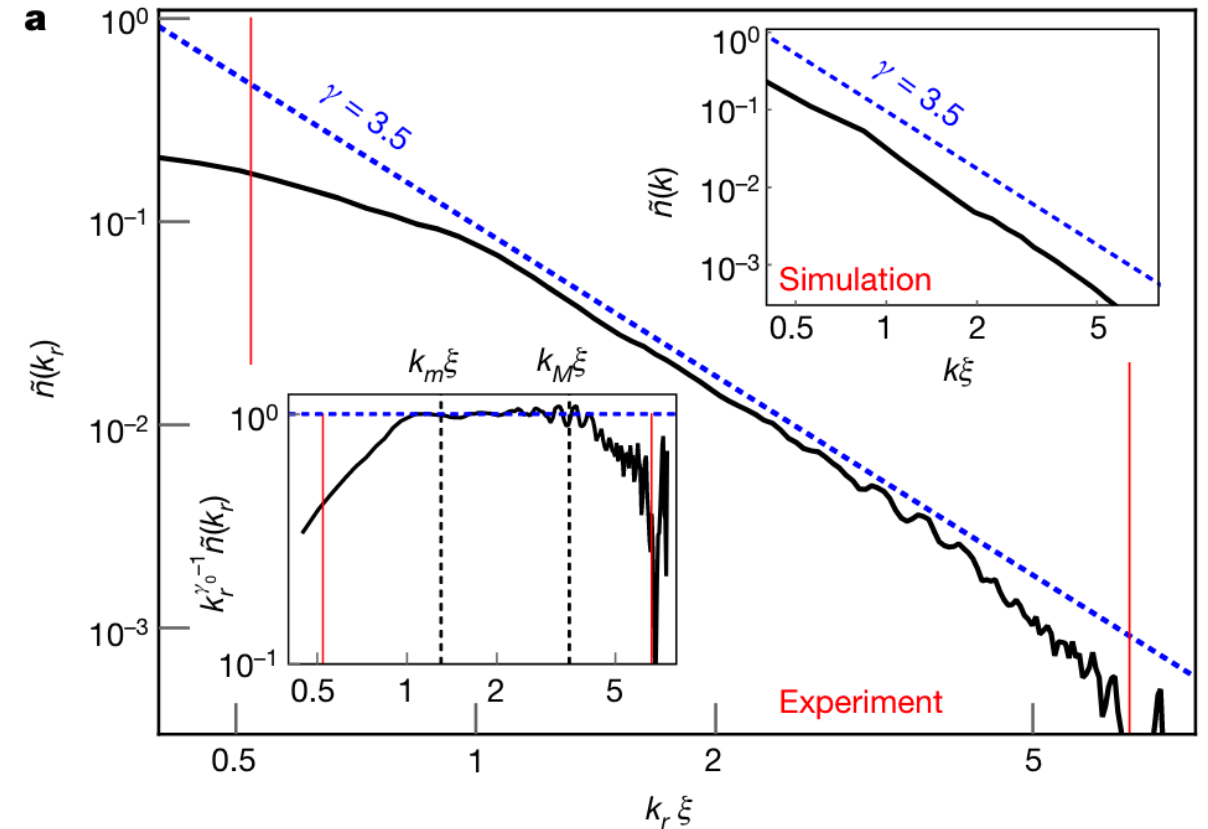
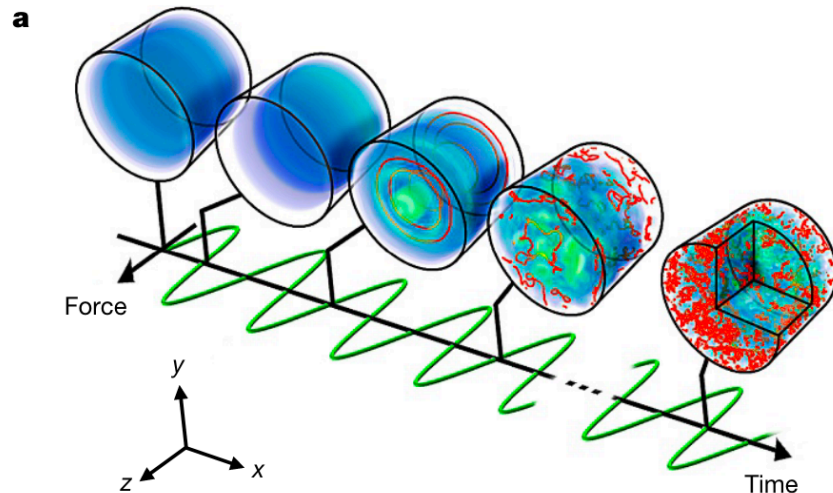
Inverse Energy Cascade



M. T. Reeves *et al*, PRL (2013).

Forced 3D turbulence

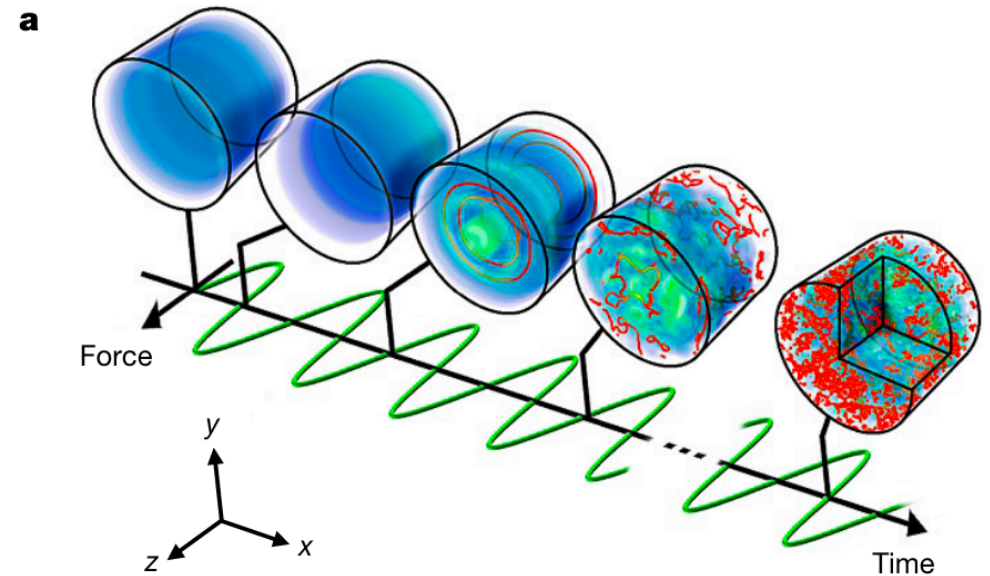
- Reported power law $k^{-3.5}$
- Narrow inertial range, in agreement with simulations
- Differed from weak wave turbulence (WWT) prediction of k^{-3} and from Kolmogorov $k^{-5/3}$



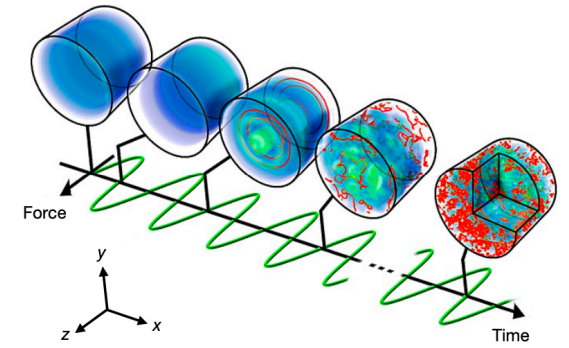
Forced 3D turbulence

Speculated difference from weak wave turbulence prediction k^{-3} due to

- Vortices
- Violation of weak wave turb. conditions
- Quantum pressure at high k



What is the mechanism of transport?



Experiment Only access the total momentum distribution $n(k)$

Theory Modern spectral methods allow accurate decomposition into vortex, sound, and quantum pressure terms, and complete account of quantum phase

Approach Large scale simulations of the GPE

Our aim: regimes

- Vary the forcing U_F , creating steady turbulence
- Novel decomposition of $n(k)$ using high resolution spectral analysis
- Identify different regimes, role of vortices, sound, quantum pressure

A. S. Bradley, R. K. Kumar, S. Pal, and X. Yu,
Spectral Analysis for Compressible Quantum Fluids,
Phys. Rev. A **106**, 043322 (2022).

Model System

Shaking a GPE ground state in a box trap

System

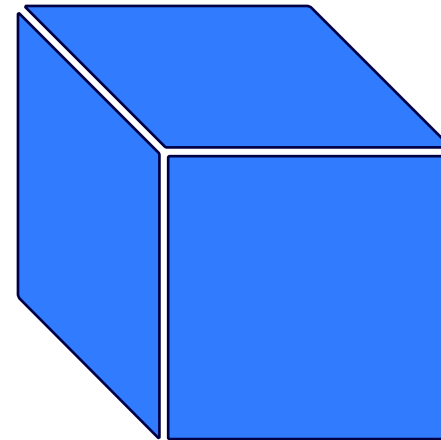
Ground state

Natural units

- Healing length ξ
- Chemical potential μ

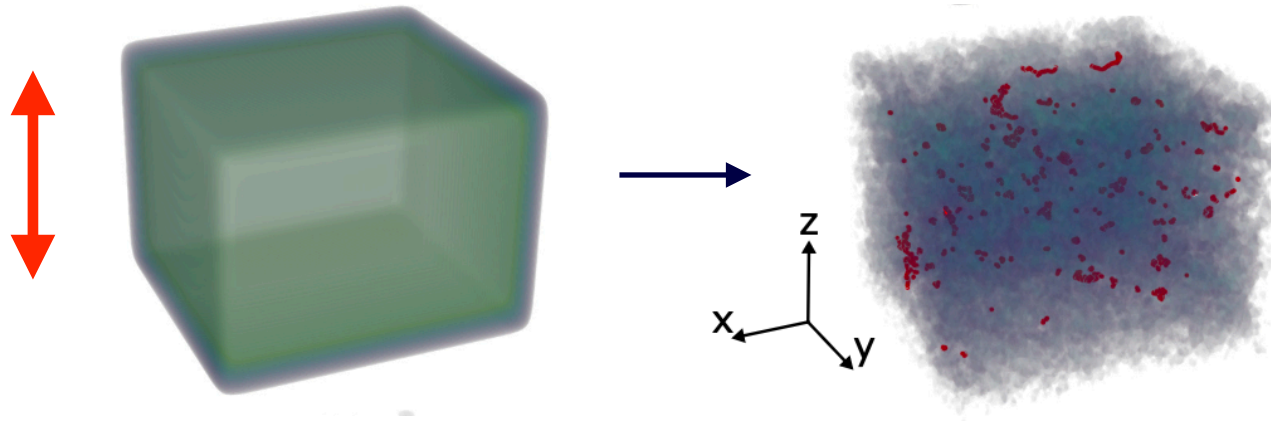
$$\mu = gn_0 \equiv \frac{\hbar^2}{m\xi^2} \equiv mc^2$$

$$i\hbar \frac{\partial \psi}{\partial t} = \left(-\frac{\hbar^2 \nabla^2}{2m} + V(\mathbf{r}, t) + g|\psi|^2 \right) \psi$$
$$= \mu \psi$$



System: creating-steady turbulence

GPE simulations



Anisotropic Shaking → **Isotropic Turbulence**

Box trap: $(L_x, L_y, L_z) = (40, 30, 20)\xi$

Grids: (256, 256, 256)

Potential height: $V_b = 30\mu$

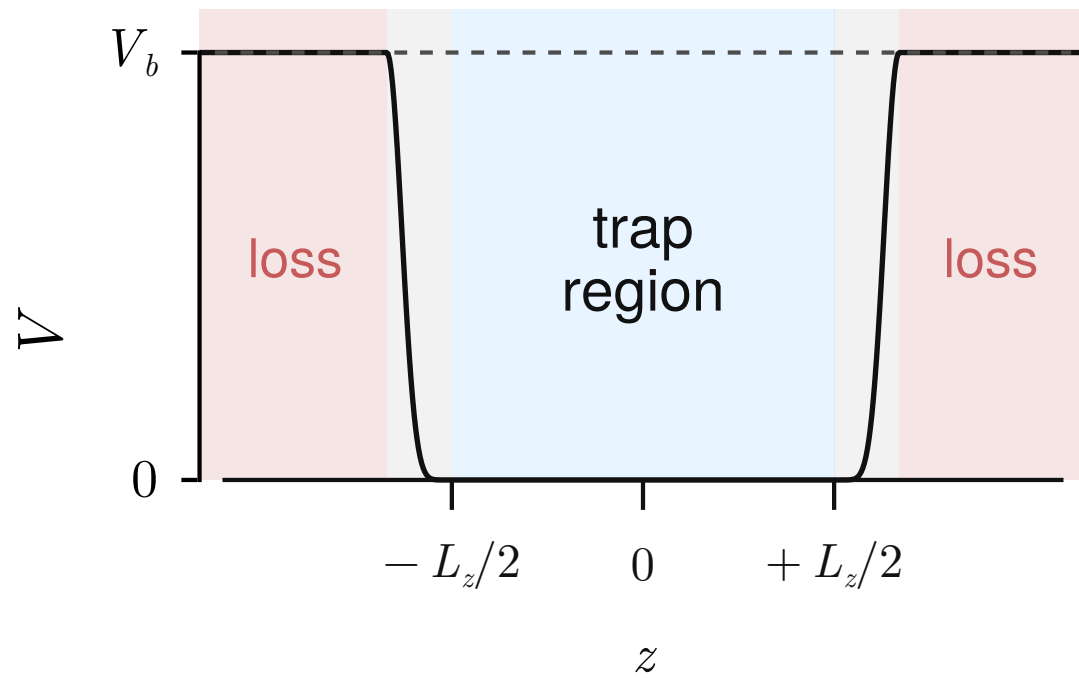
Forcing: $V_F(\mathbf{r}, t) = U_F \frac{z}{L_z} \sin(\omega_F t)$

Amplitude range: $0.1\mu \leq U_F \leq 5\mu$

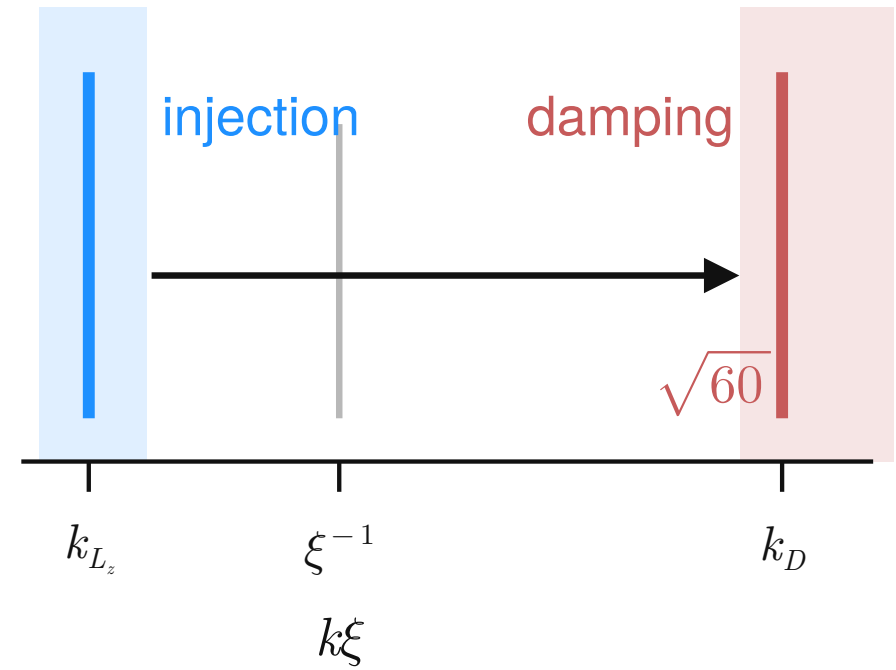
Dissipation: Atoms have to scale the walls to energy V_b , with dissipation scale $k_D \xi = \sqrt{2V_b/\mu} = 7.7$

System: energies and scales

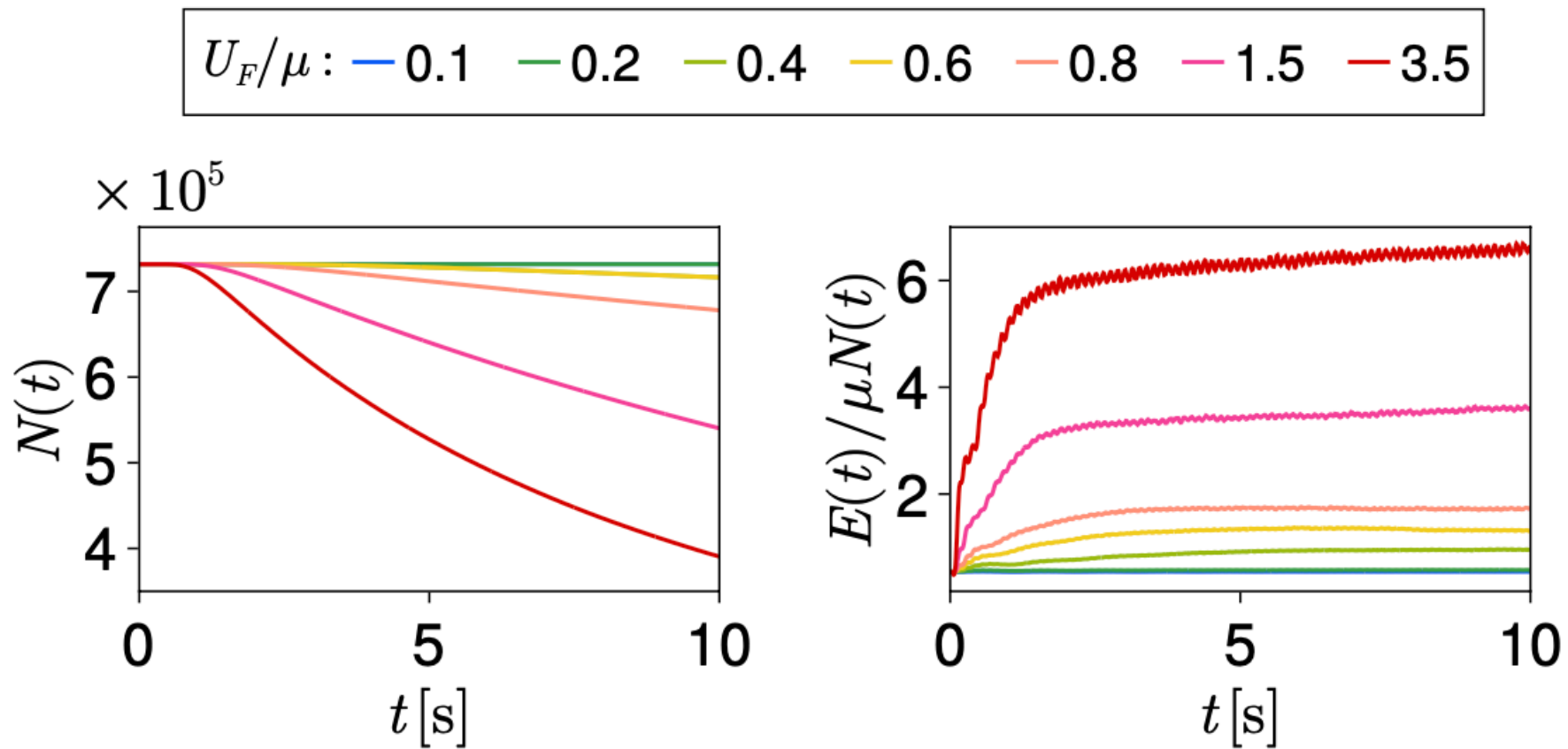
real space



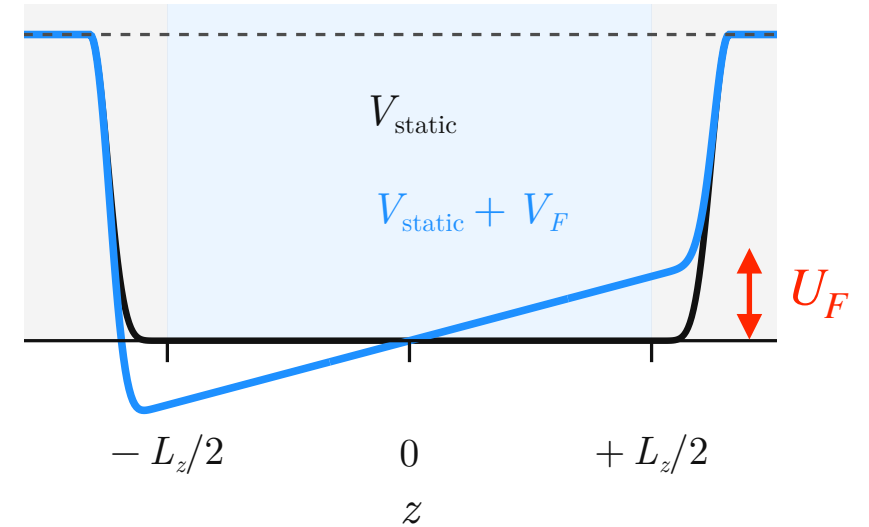
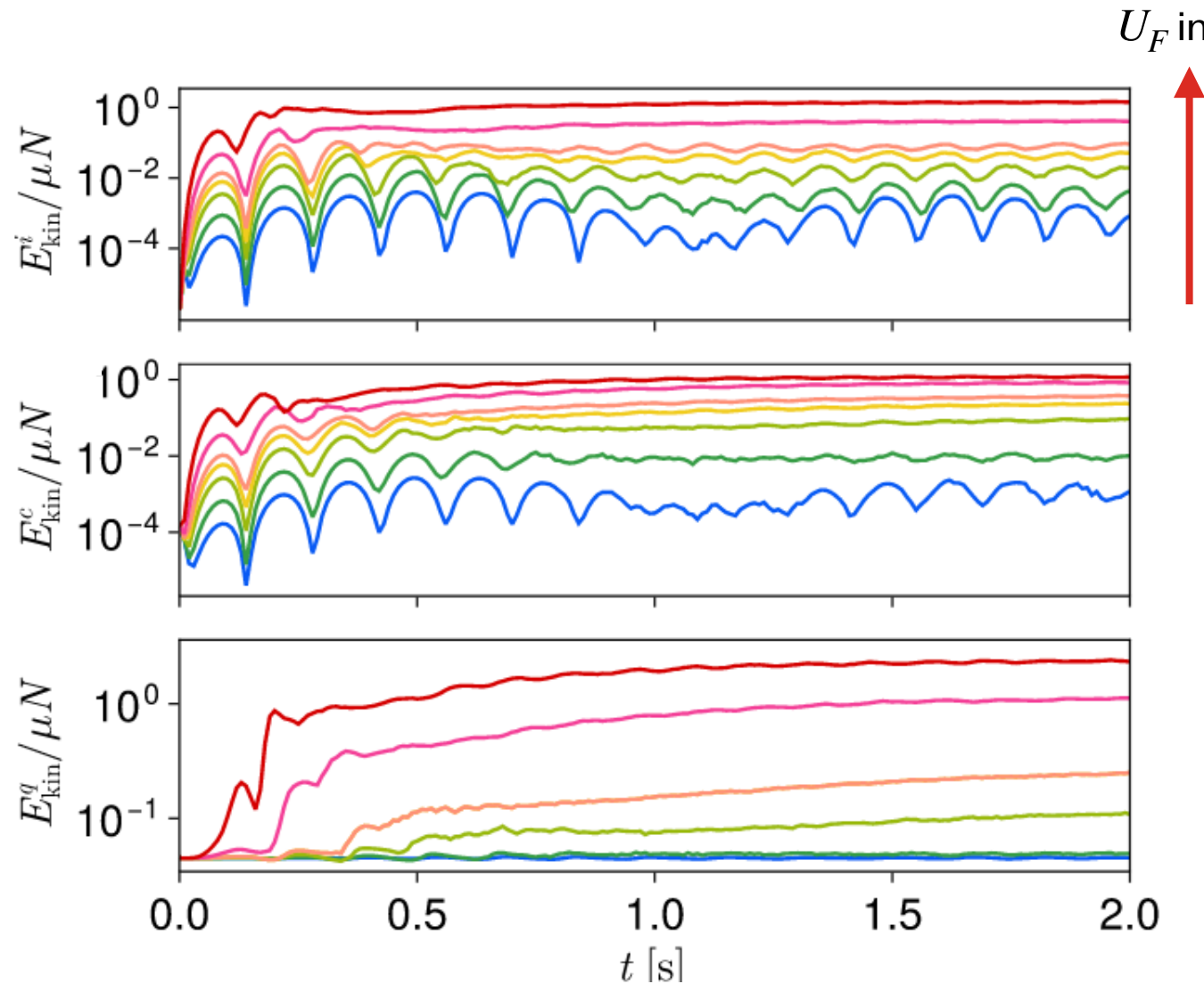
wavenumber space



Dynamics



Driven system reaches a statistically steady state

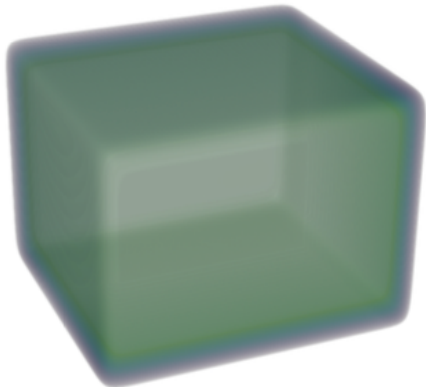


Continuous forcing:

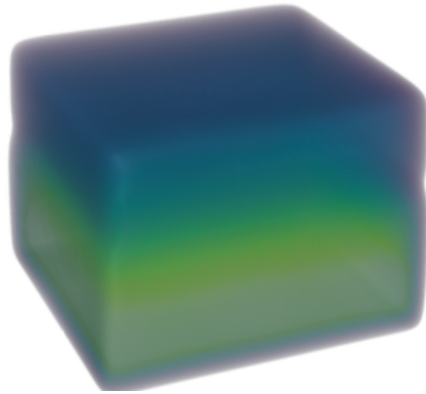
$$V_F(\mathbf{r}, t) = U_F \frac{z}{L_z} \sin(\omega_F t)$$

Dynamics

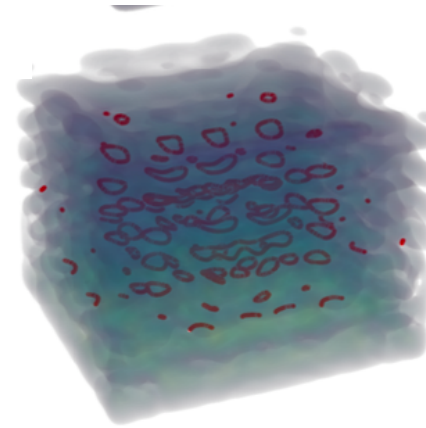
Towards steady state for $U_F/\mu = 1$



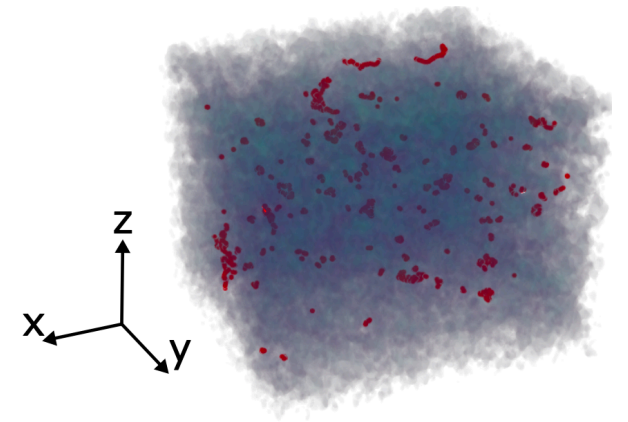
Ground
state



Density
waves



Vortex
rings



Isotropic
turbulence

Kinetic energy

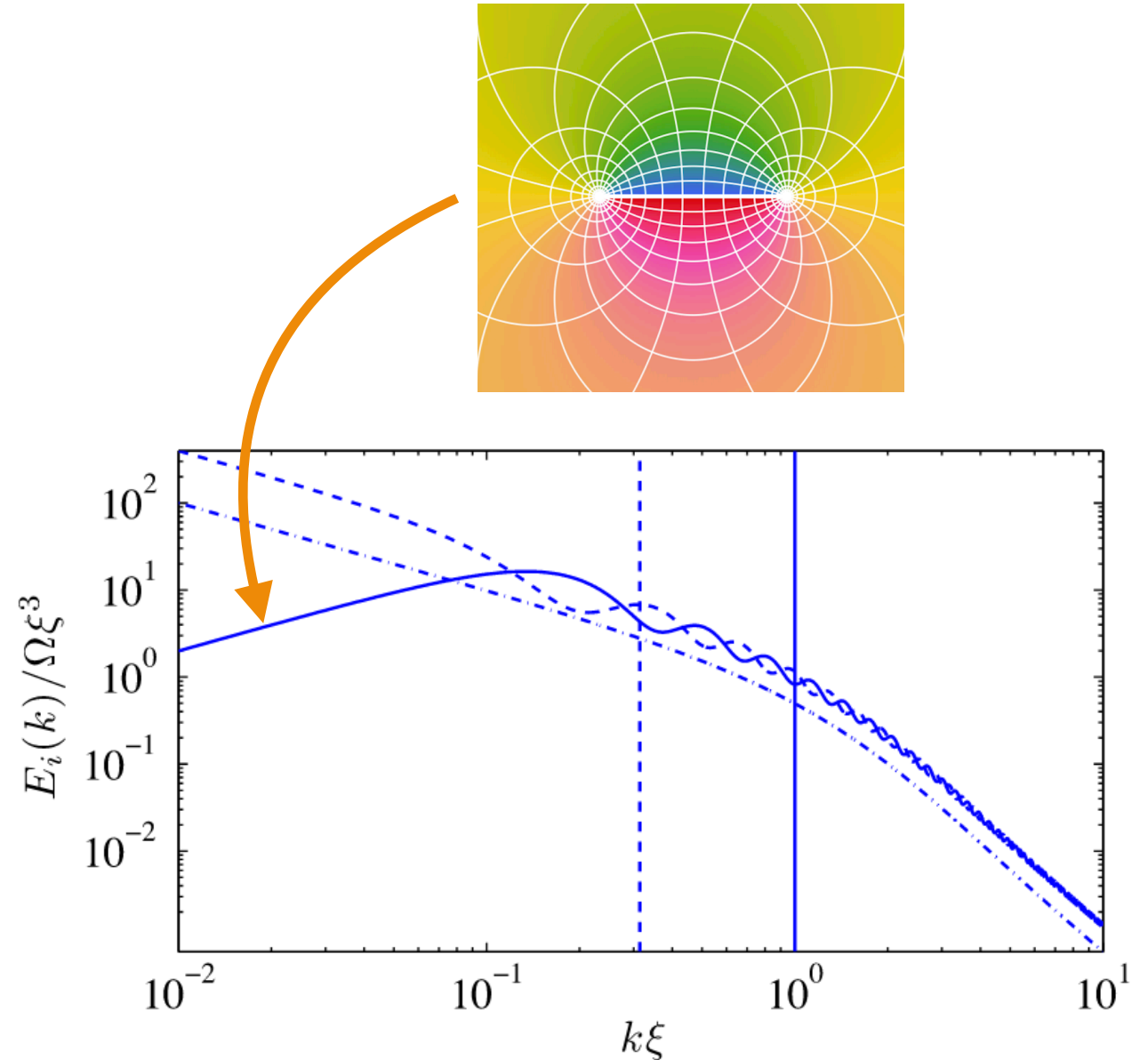
$$\psi = \sqrt{n} e^{i\Theta} \implies \mathbf{v} = (\hbar/m) \nabla \Theta$$

- Apply Helmholtz decomposition to the field

$$\sqrt{n} \mathbf{v} \equiv \mathbf{u} = \mathbf{u}^i + \mathbf{u}^c$$

where $\nabla \cdot \mathbf{u}^i = 0$
 $\nabla \times \mathbf{u}^c = 0$

- Neglects quantum phase information.

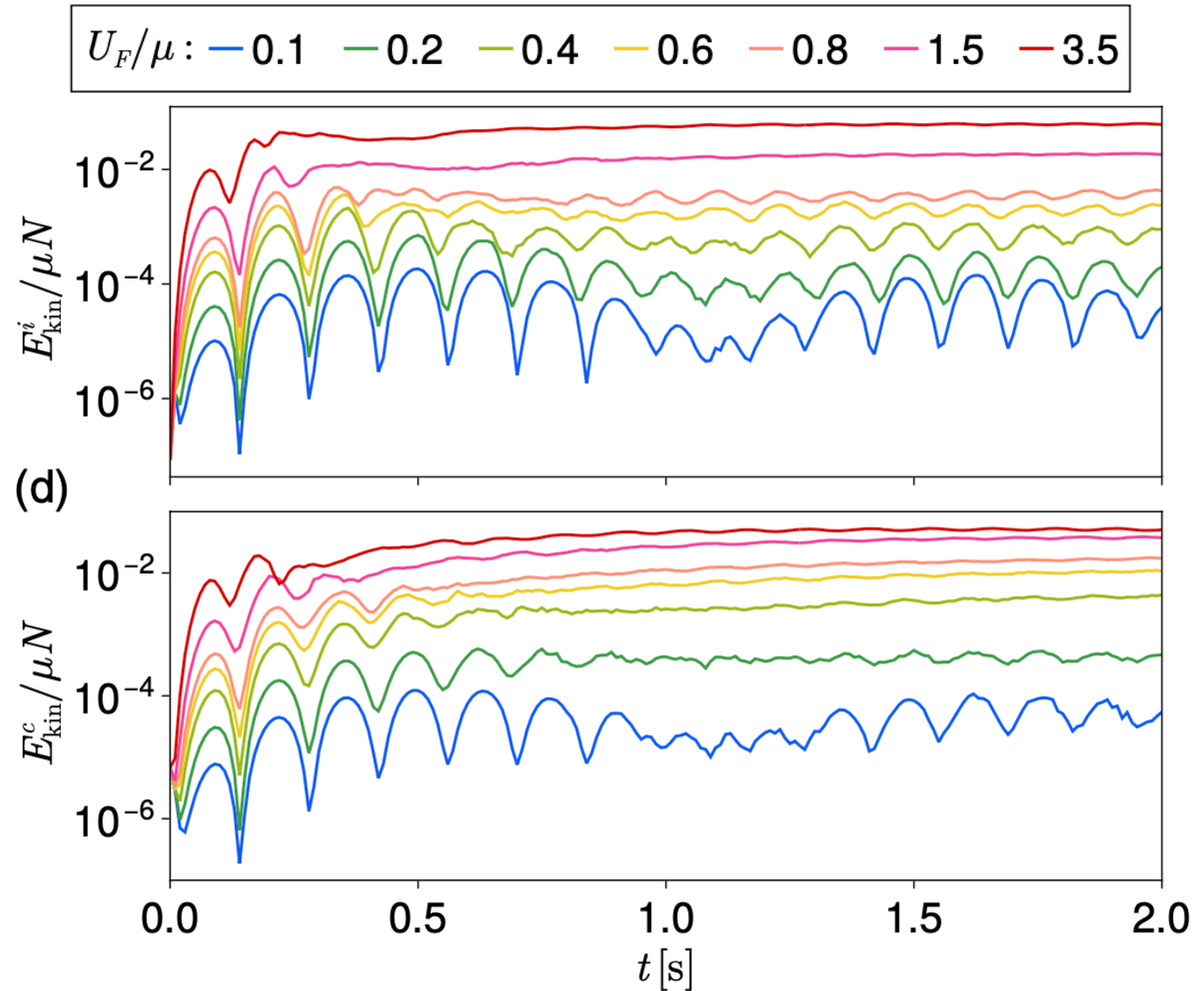


Dynamics

- $E_{\text{kin}}^i = \frac{m}{2} \int d^3\mathbf{r} |\mathbf{u}^i|^2$

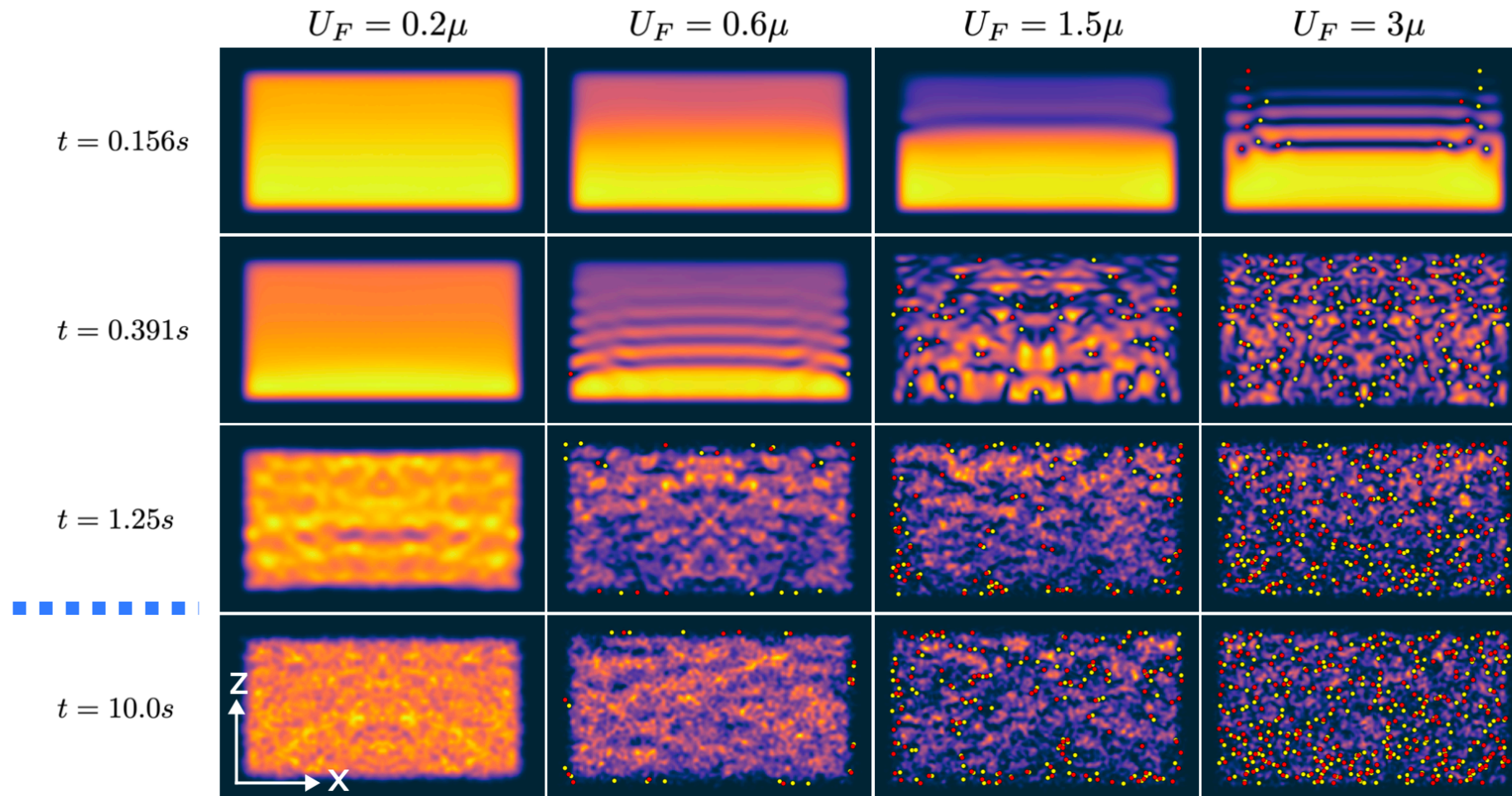
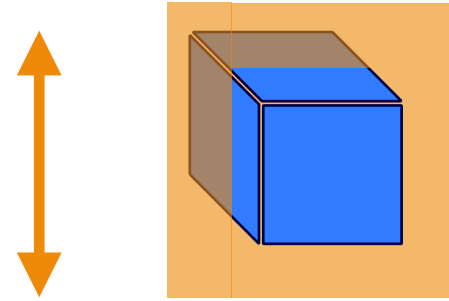
- $E_{\text{kin}}^c = \frac{m}{2} \int d^3\mathbf{r} |\mathbf{u}^c|^2$

- Transition from cyclic response, to noncyclic



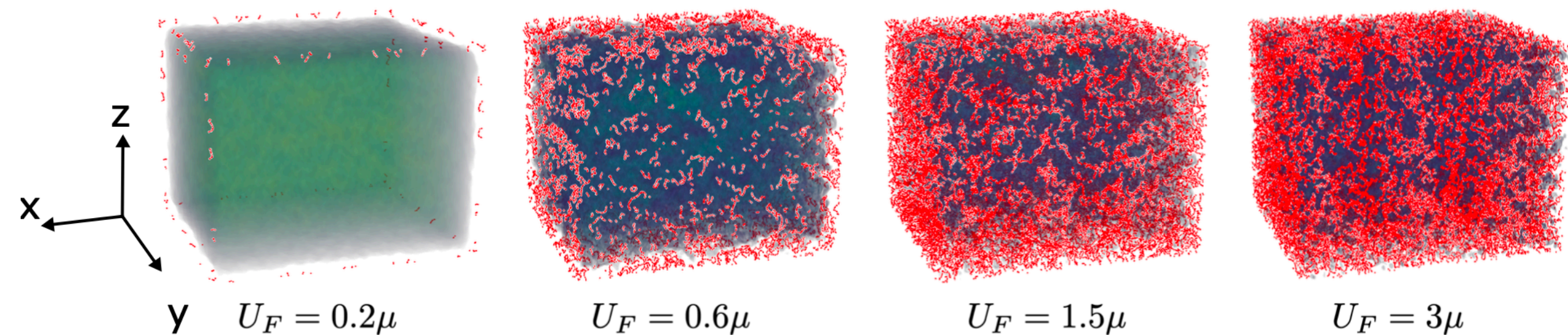
Dynamics

Vortices appear for higher μ



Steady state vortices

3D vorticity

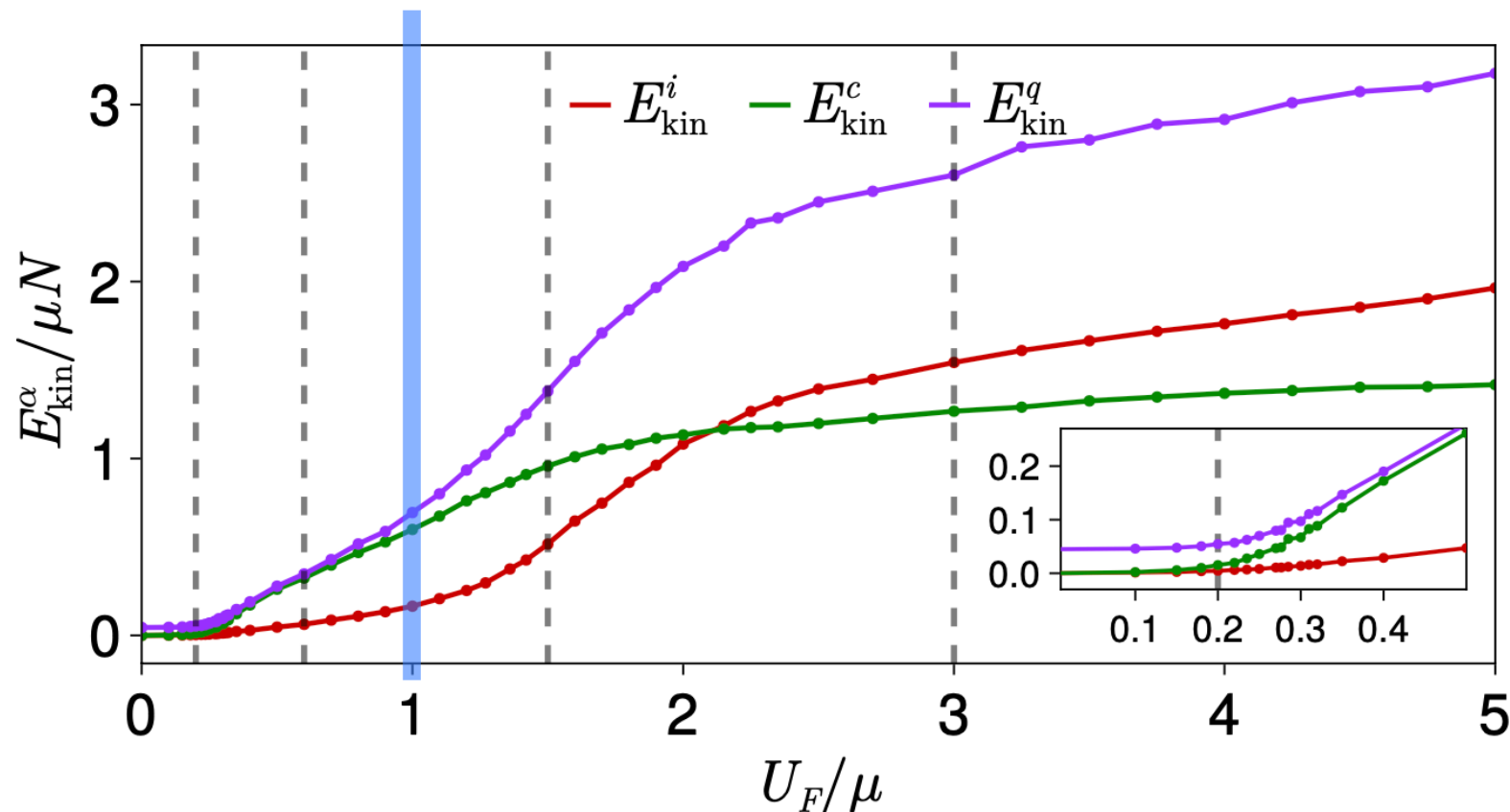


Steady state energies

Breakdown of WWT

$$U_F/\mu = 1:$$

- Quantum pressure starts to grow faster
- Vortices enter the bulk superfluid

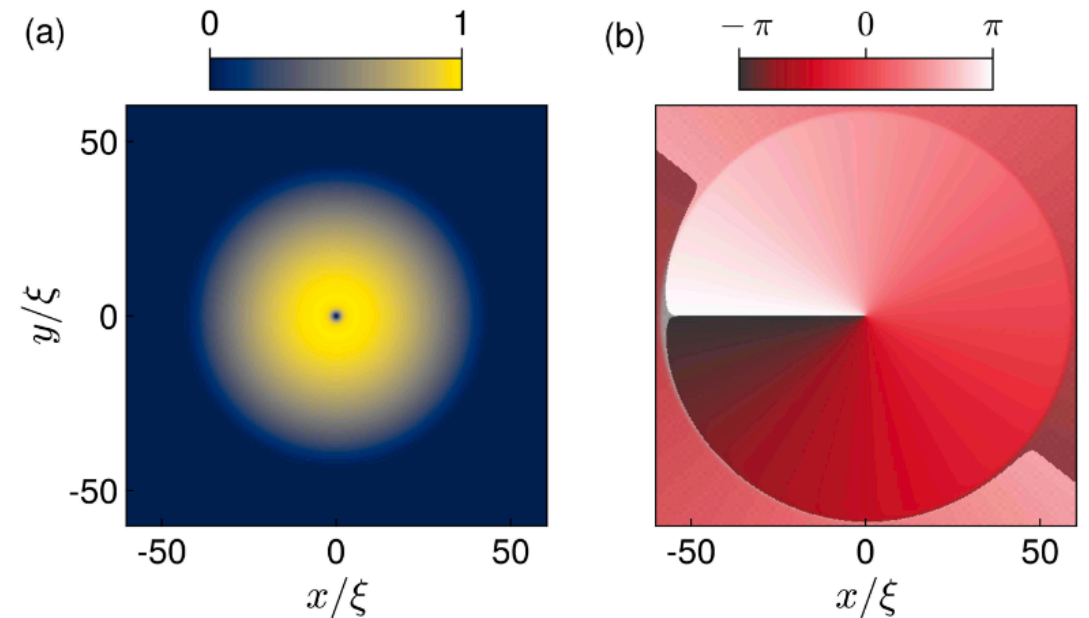


Analysis

Finding (hidden) power laws

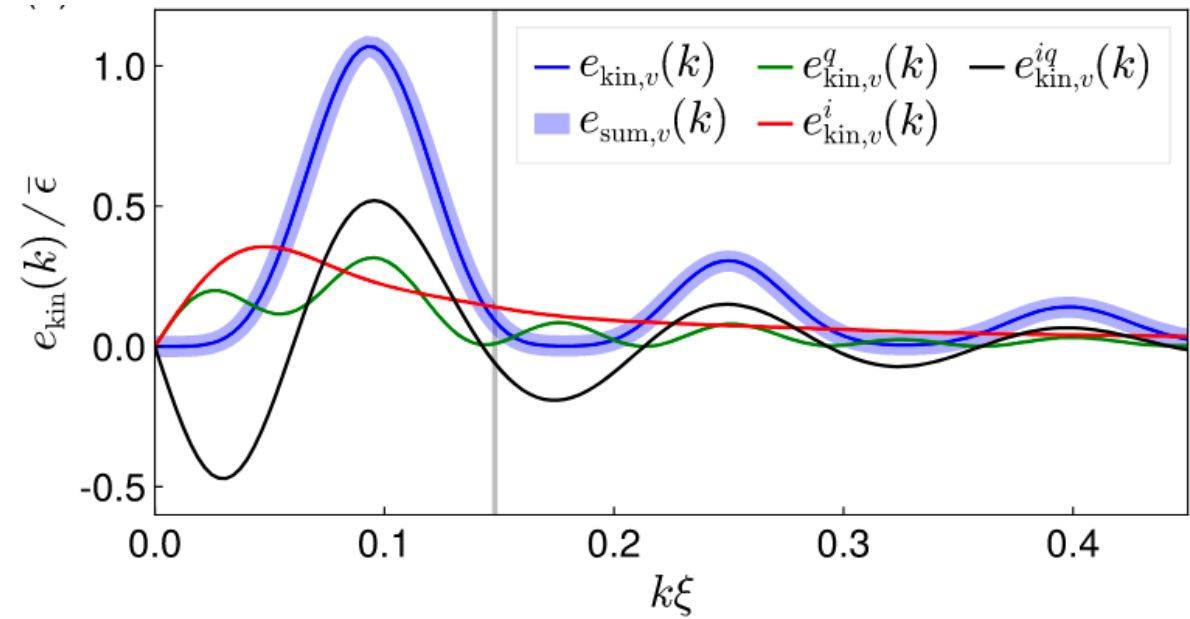
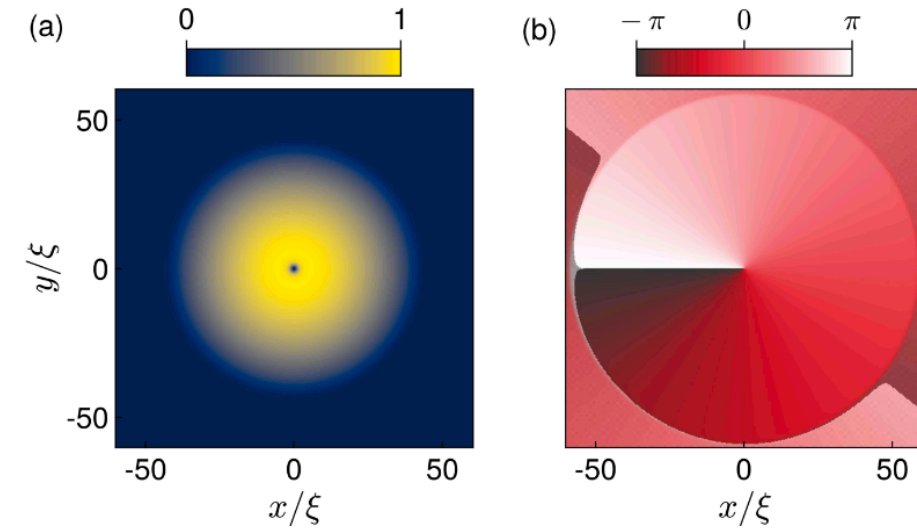
Spectral analysis: including quantum phase

- Helmholtz decomposition $\sqrt{n}\mathbf{v} \equiv \mathbf{u} = \mathbf{u}^i + \mathbf{u}^c$
- Quantum pressure: $\mathbf{u}^q = (\hbar/m) \nabla \sqrt{n}$
- Approach allows choice of k



Spectral analysis: quantum phase

- Quantum phase:
 $(\mathbf{w}^i, \mathbf{w}^c, \mathbf{w}^q) = (i\mathbf{u}^i, i\mathbf{u}^c, \mathbf{u}^q)e^{i\Theta}$.
- Then: $(\hbar/m) \nabla \psi \equiv \mathbf{w}^i + \mathbf{w}^c + \mathbf{w}^q$
- True energy densities in Fourier space:
 $|\tilde{\mathbf{w}}^i(\mathbf{k})|^2$



Momentum distribution

Basic idea: don't do binning in Fourier space to make spectra

$$N = \int d^3\mathbf{r} |\psi|^2 = \int d^3\mathbf{k} |\tilde{\psi}(\mathbf{k})|^2 \equiv \int_0^\infty dk 4\pi k^2 n(k),$$

where angular integral is analytic:

$$n(k) = \frac{1}{(2\pi)^3} \int d^3\mathbf{r} \text{sinc}(k|\mathbf{r}|) C[\psi, \psi](\mathbf{r}), \quad (\text{without assuming isotropy}),$$

$$C[\psi, \psi](\mathbf{r}) = \int d^3\mathbf{R} \psi^*(\mathbf{R} - \mathbf{r}/2) \psi(\mathbf{R} + \mathbf{r}/2), \text{ two point correlation (DFT).}$$

Momentum distribution decomposition

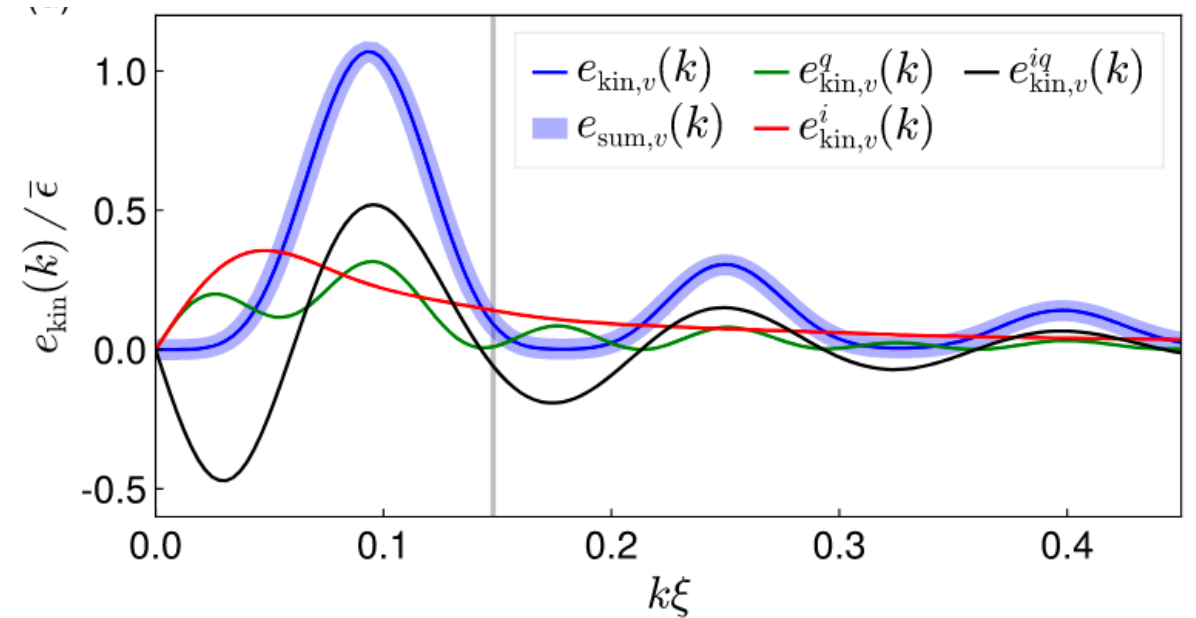
Same approach as for $n(k)$, applied to energy density corresponding to

$$(\hbar/m) \nabla \psi = \mathbf{w}^i + \mathbf{w}^c + \mathbf{w}^q$$

$$\begin{aligned} e_{\text{kin}}(k) &= \frac{\hbar^2 k^2}{2m} n(k) \\ &= e_{\text{kin}}^c(k) + e_{\text{kin}}^i(k) + e_{\text{kin}}^q(k) + \text{cross corr. from } C[\mathbf{w}^\alpha, \mathbf{w}^\beta] \end{aligned}$$

Decomposed momentum distribution into

$$\text{components } n^\alpha(k) = \frac{2m e_{\text{kin}}^\alpha(k)}{\hbar^2 k^2}$$



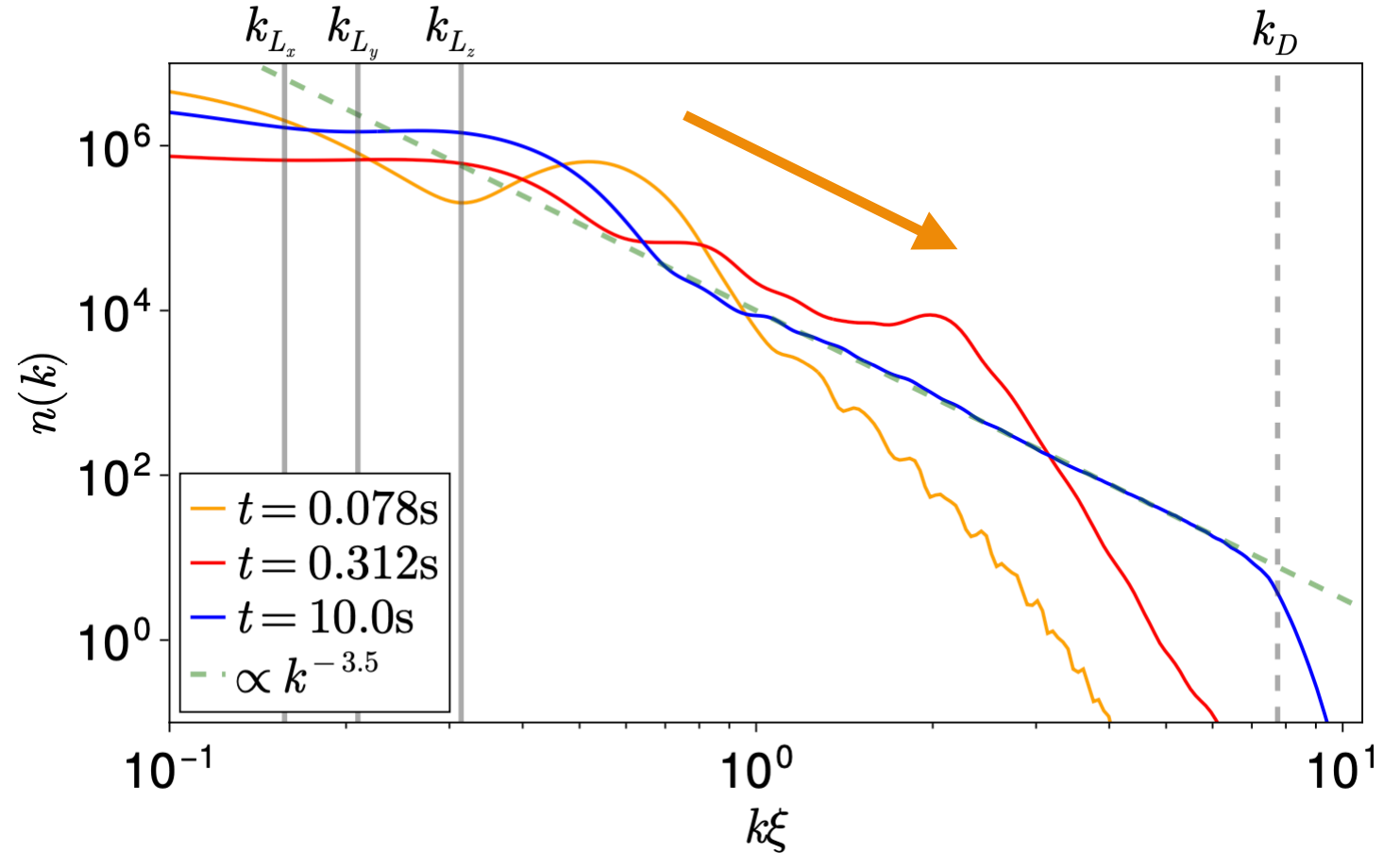
Momentum distribution

Response to forcing

Forcing peak near k_{L_z}

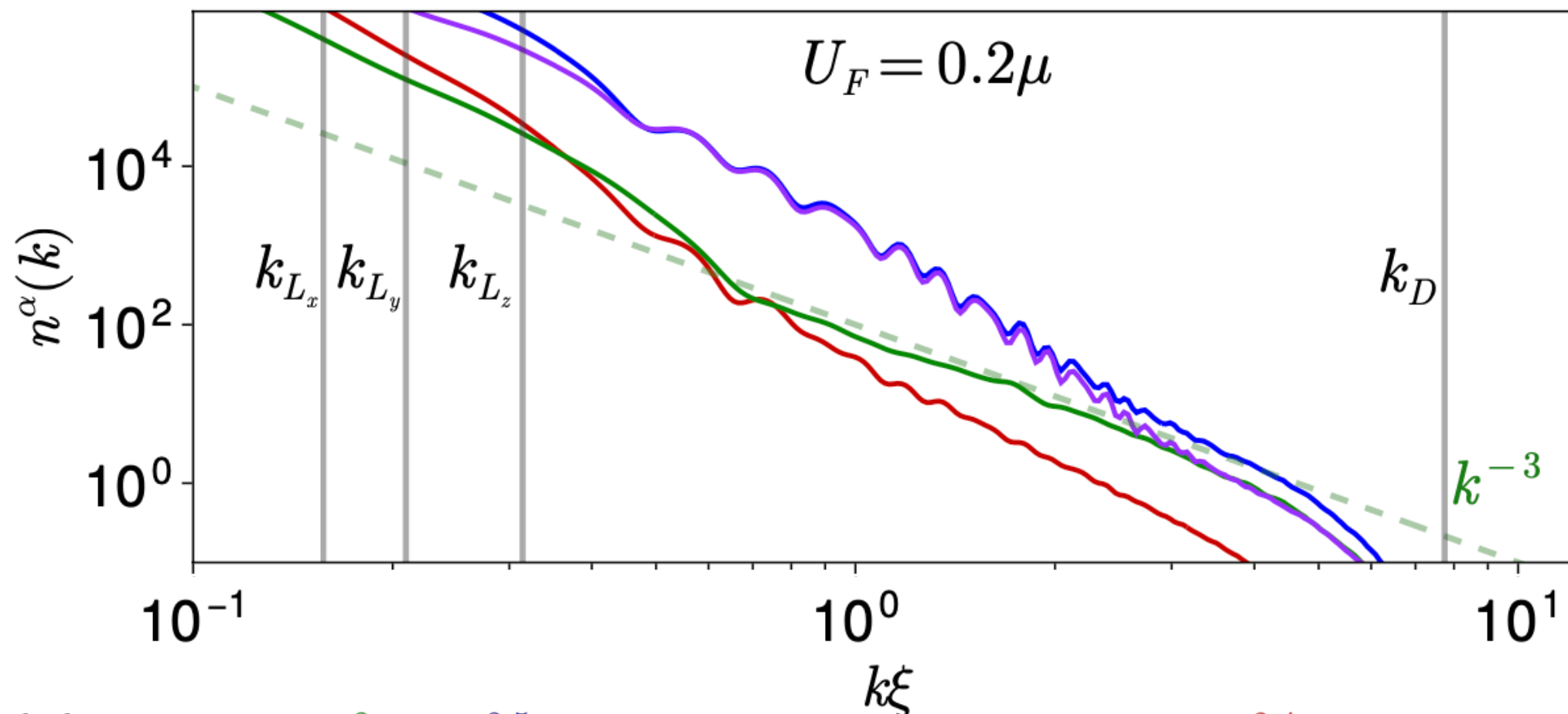
Transport to high k

Establish power law



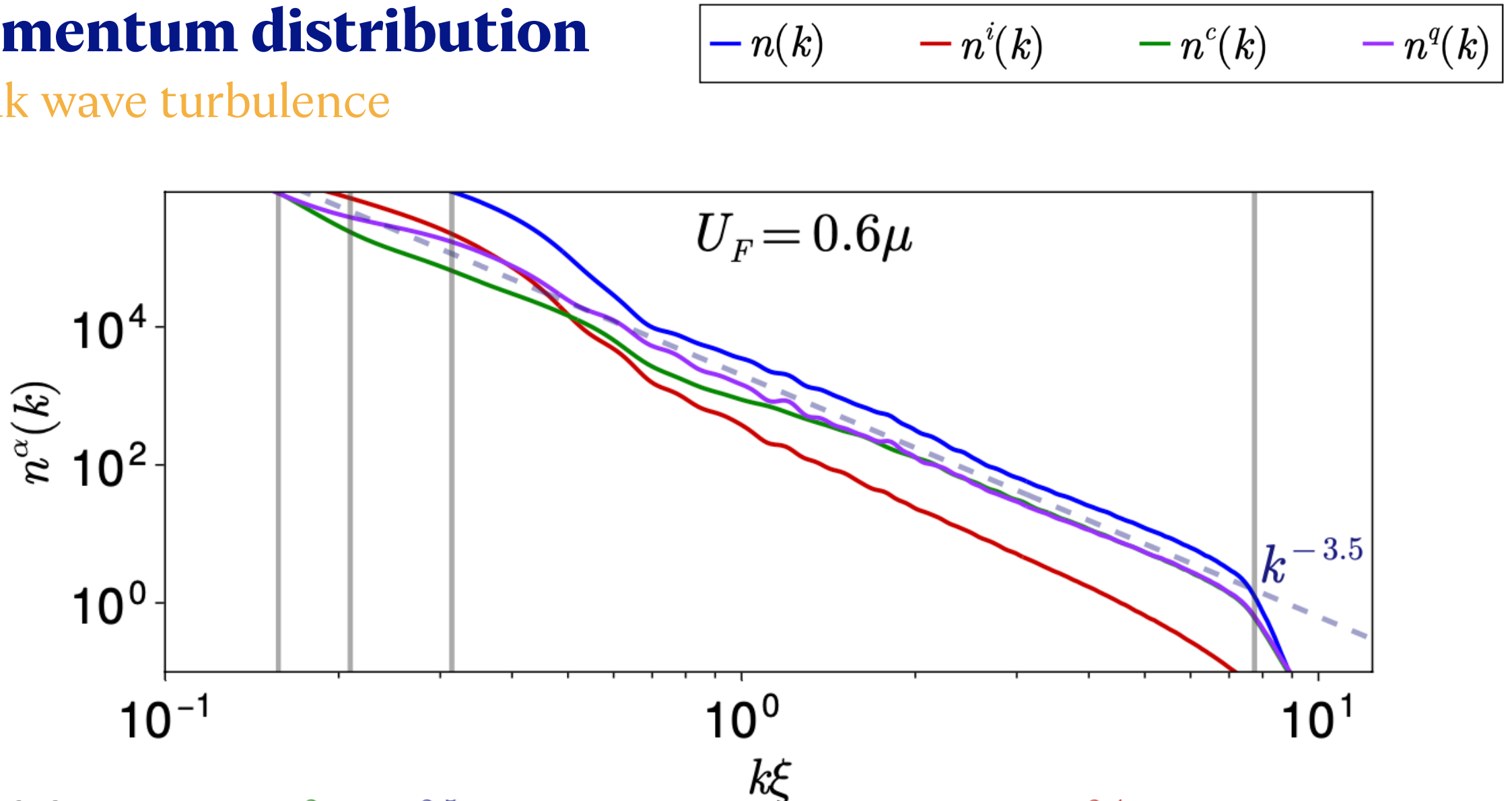
Momentum distribution

Wave turbulence



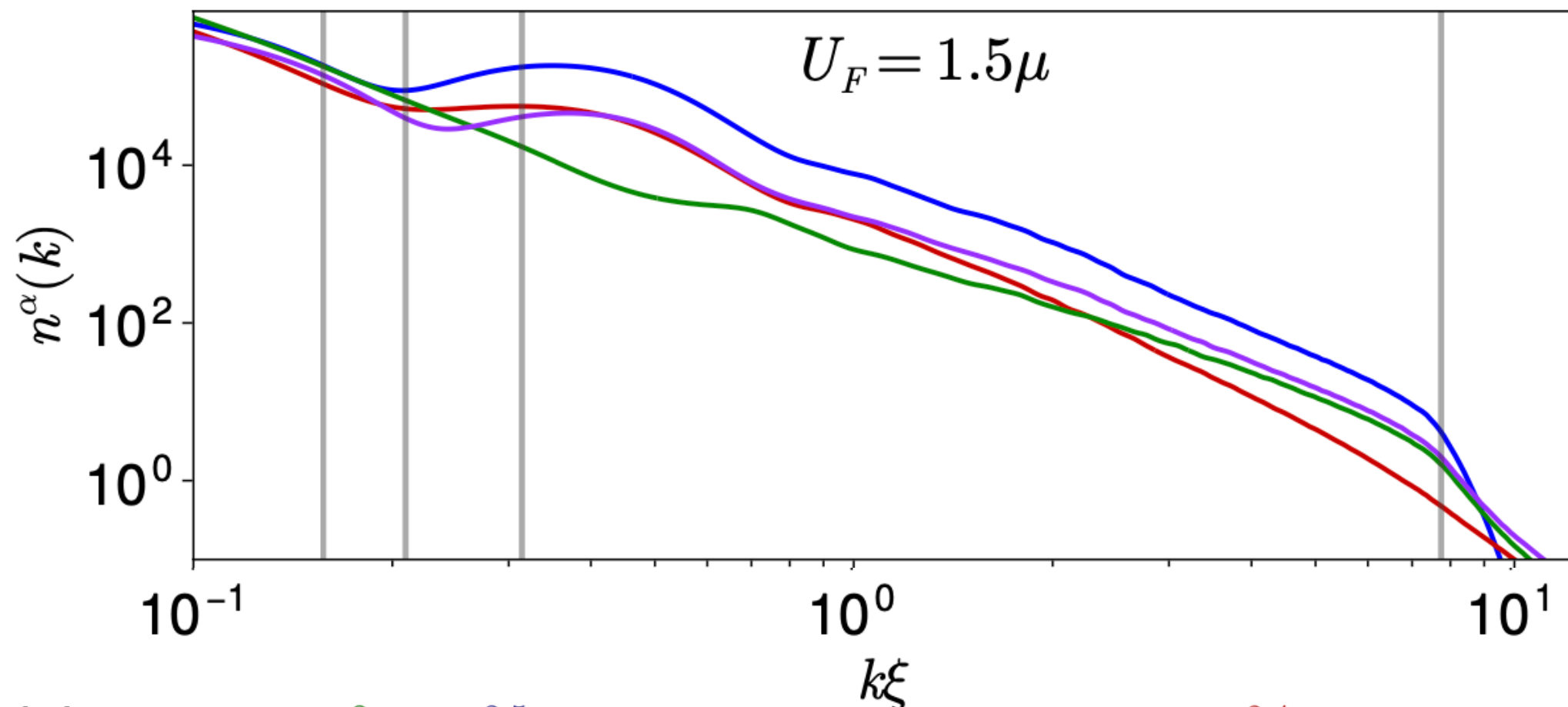
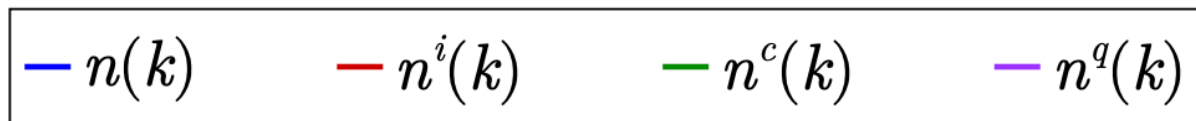
Momentum distribution

Weak wave turbulence



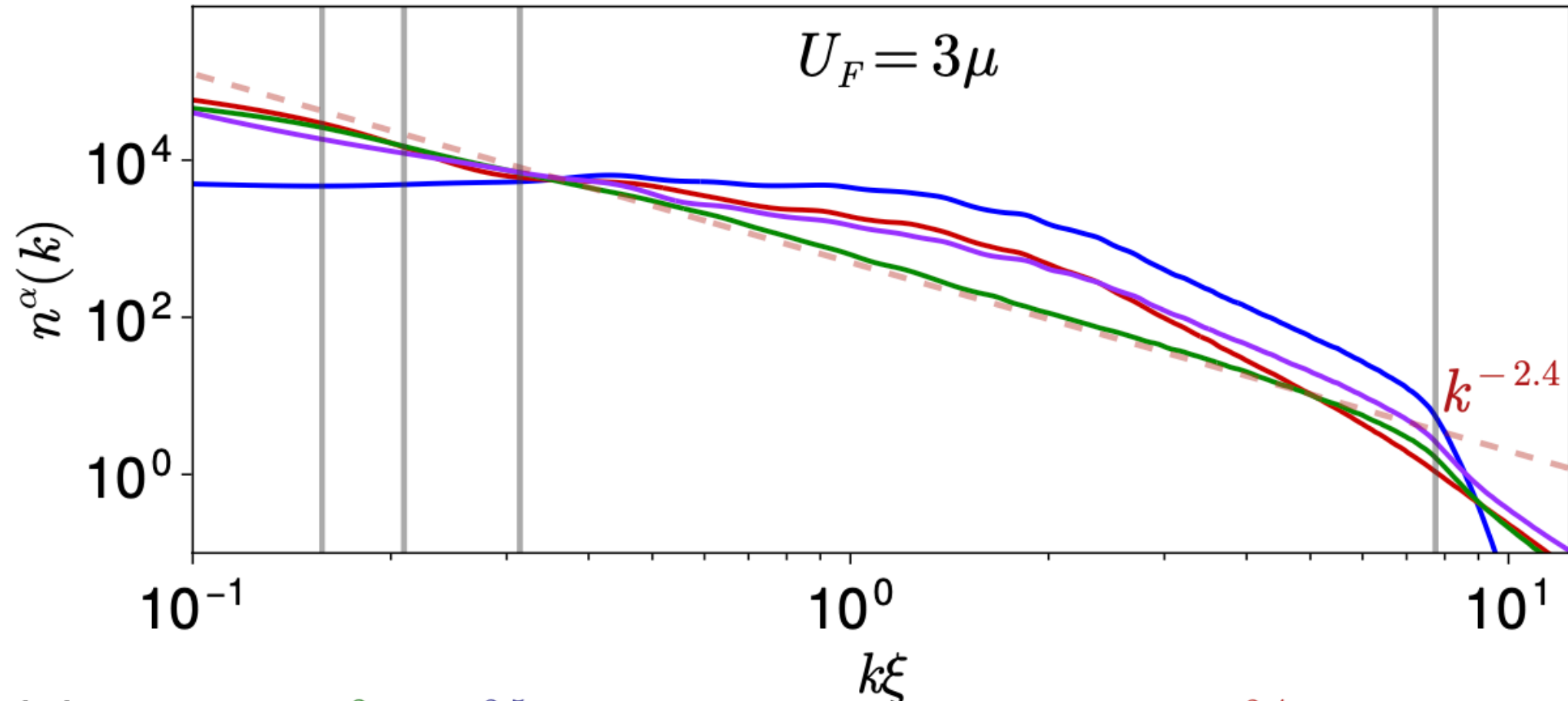
Momentum distribution

Vortex turbulence

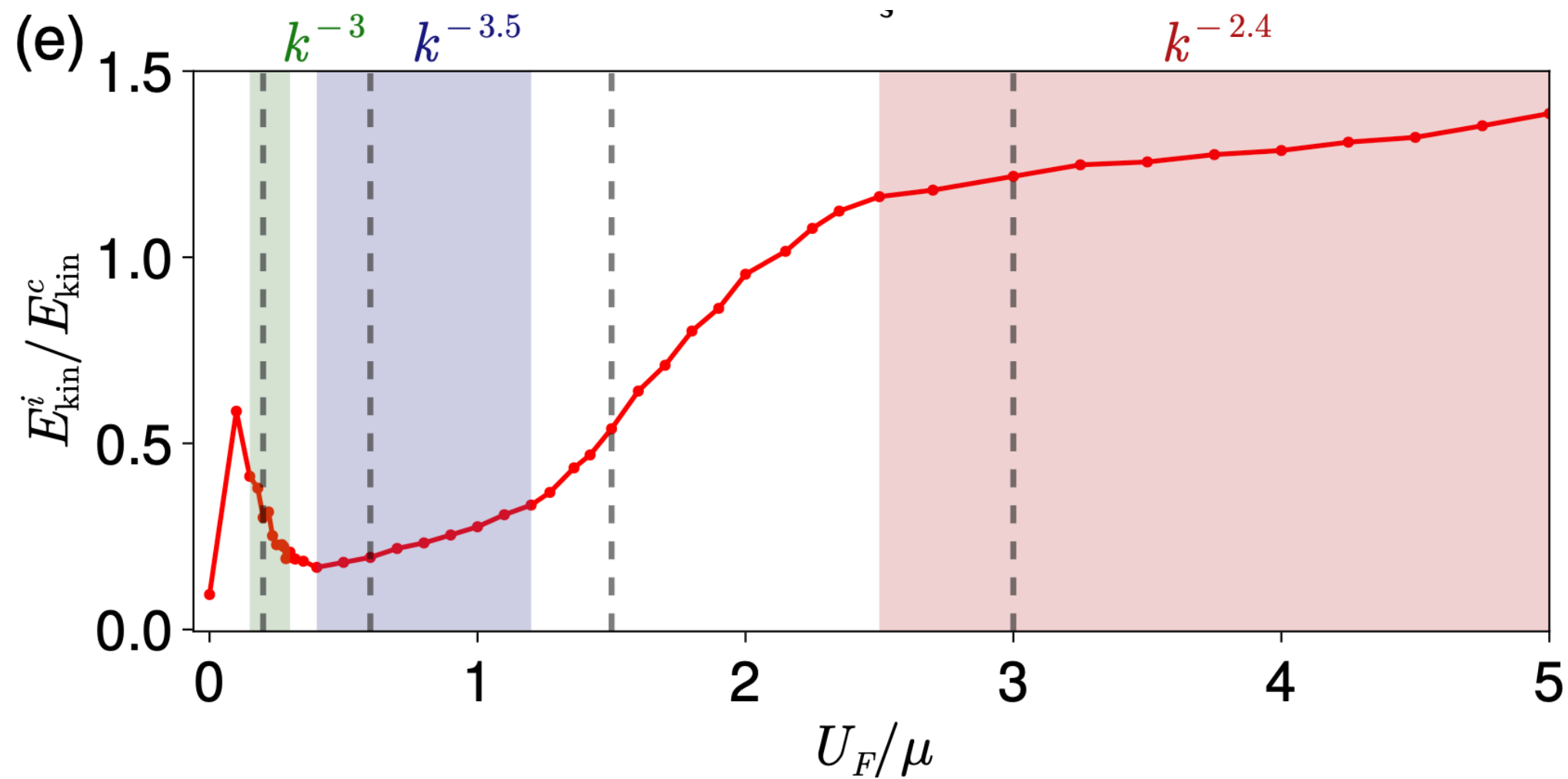


Momentum distribution

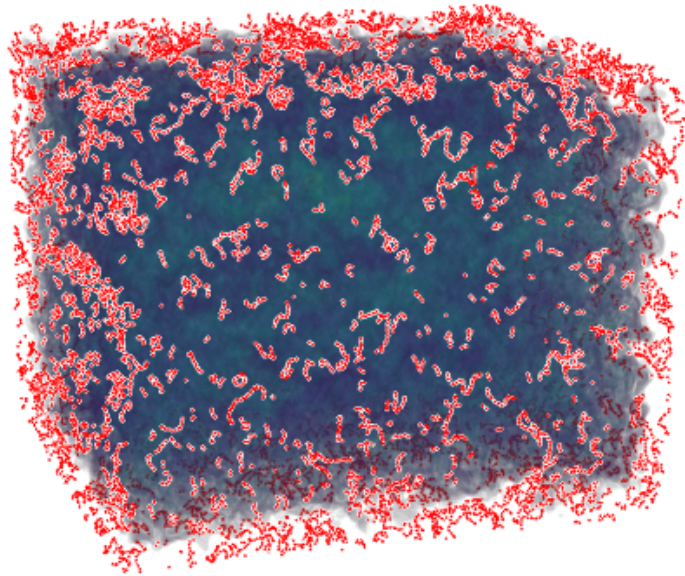
Strong vortex turbulence



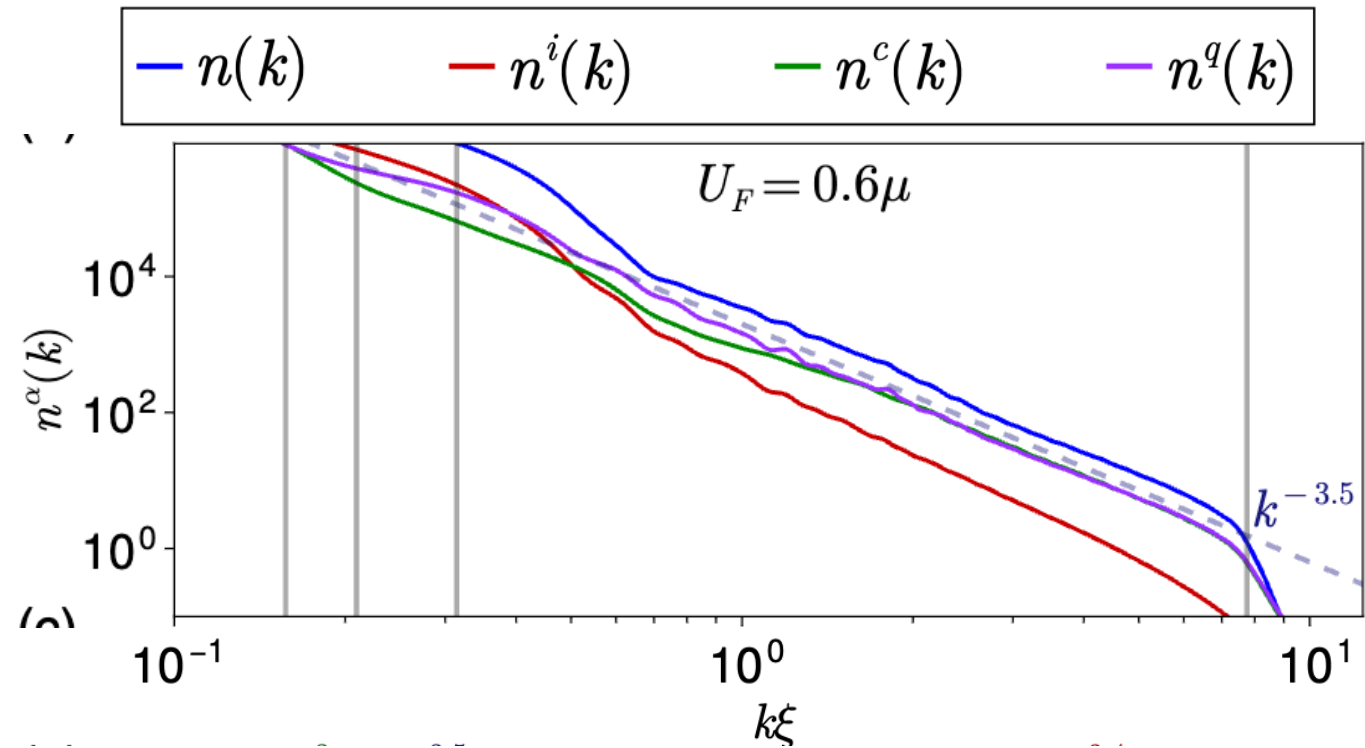
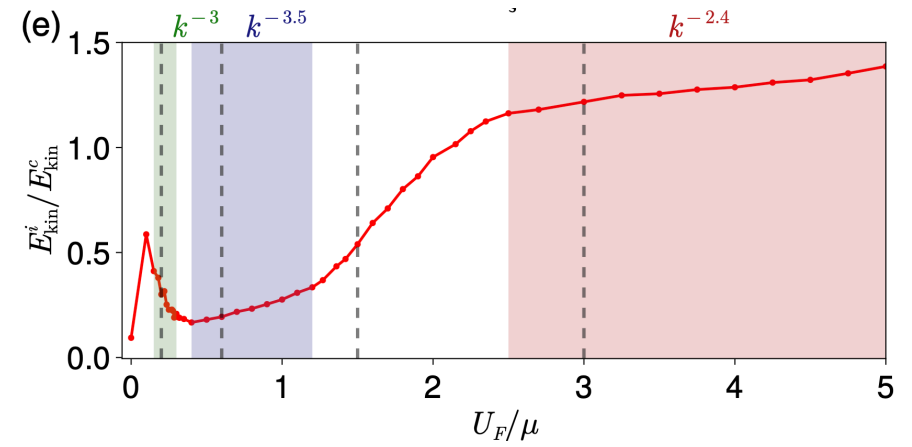
Regimes



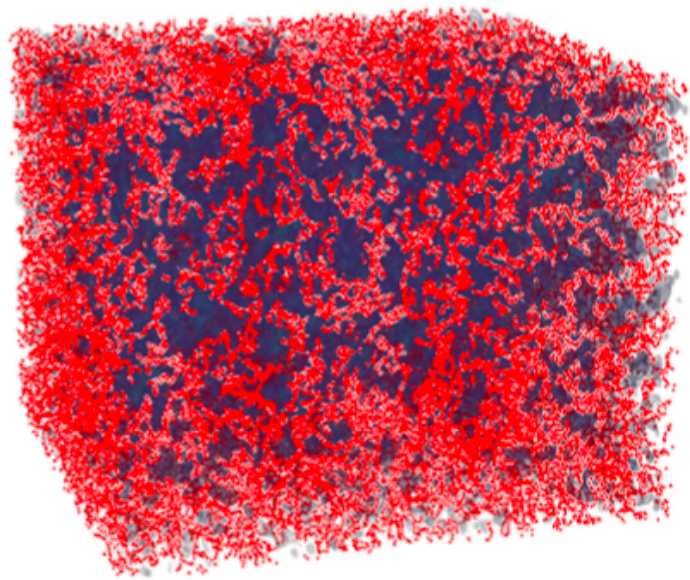
Weak wave turbulence



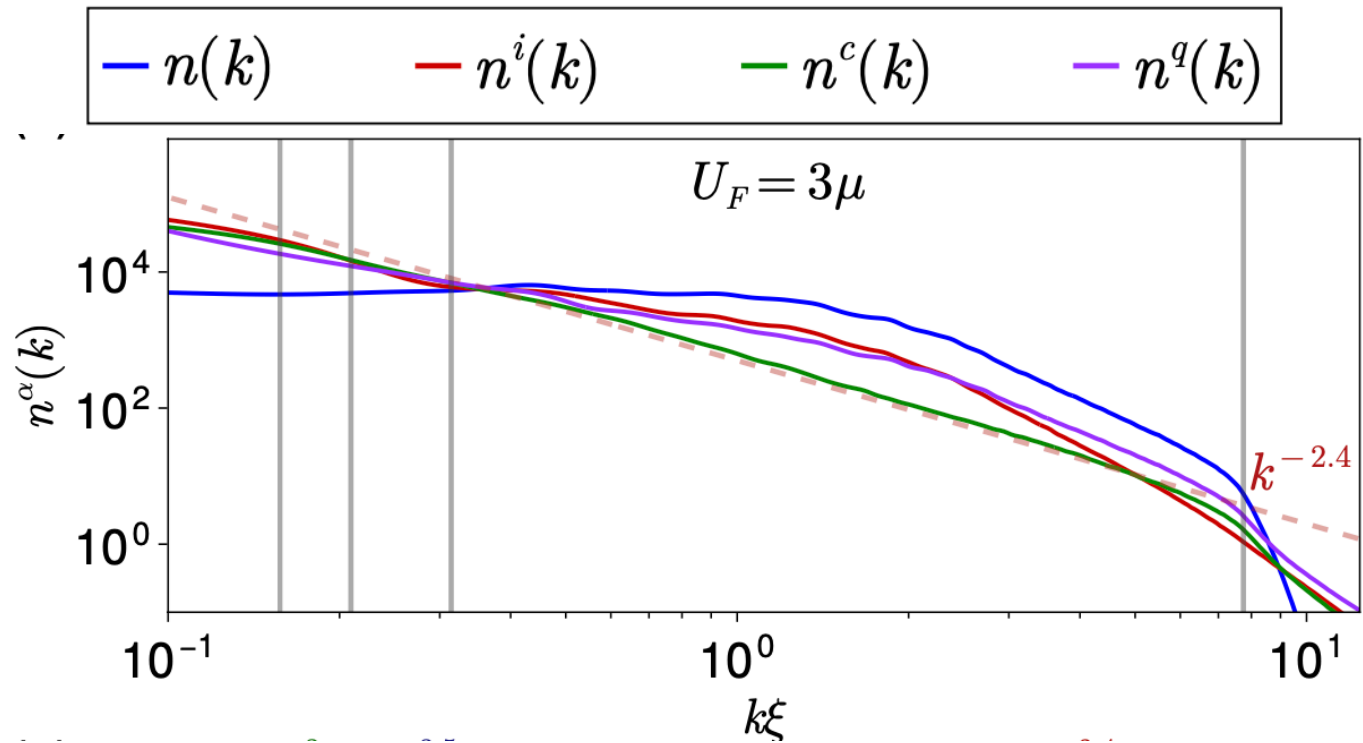
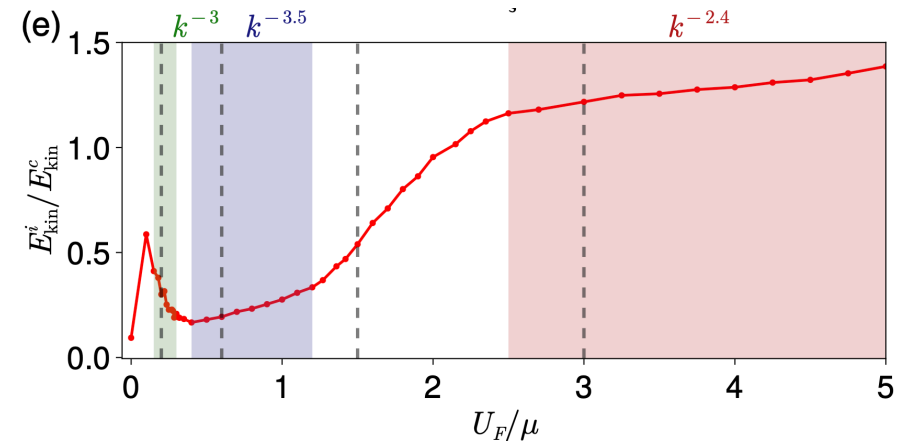
$$U_F = 0.6\mu$$



Strong vortex turbulence



$$U_F = 3\mu$$

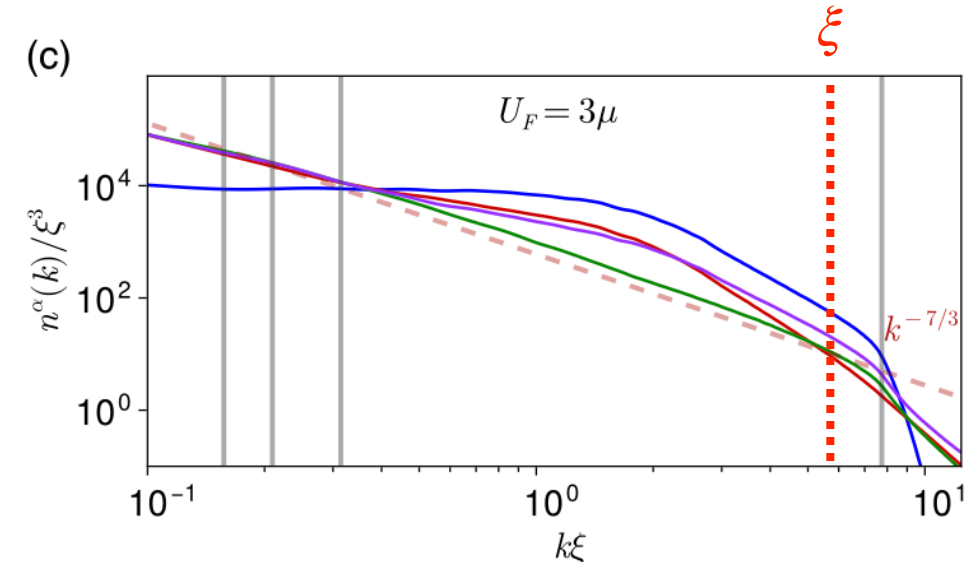
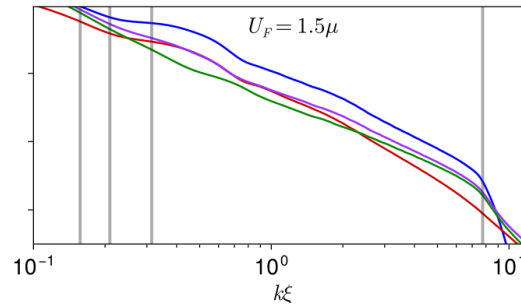
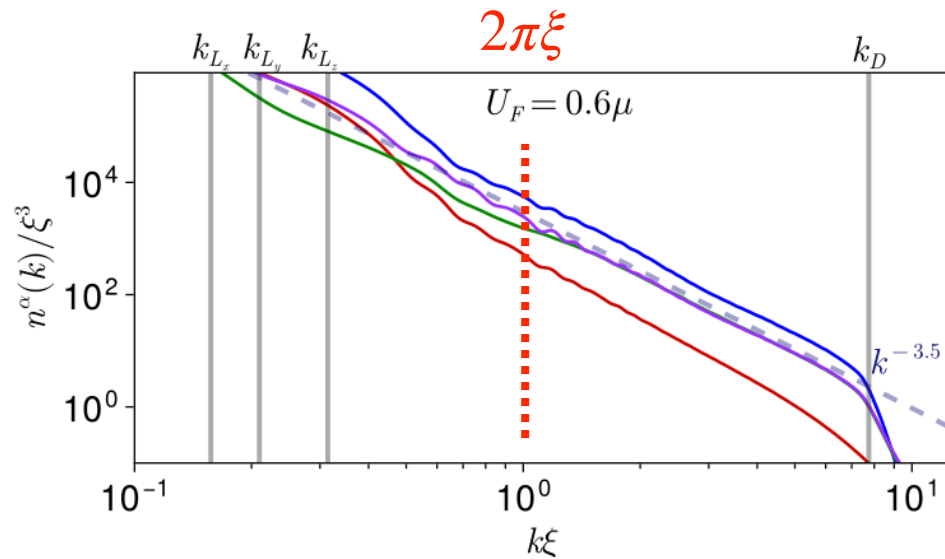


Decomposing the spectrum

$$\text{--- } n(k) \quad \text{--- } n^i(k) \quad \text{--- } n^c(k) \quad \text{--- } n^q(k)$$

$$n(k) \sim n^c(k) \approx n^q(k) \sim k^{-7/2}$$

$$n^c(k) \sim k^{-7/3}$$



4-wave, isotropic, **direct energy cascade**

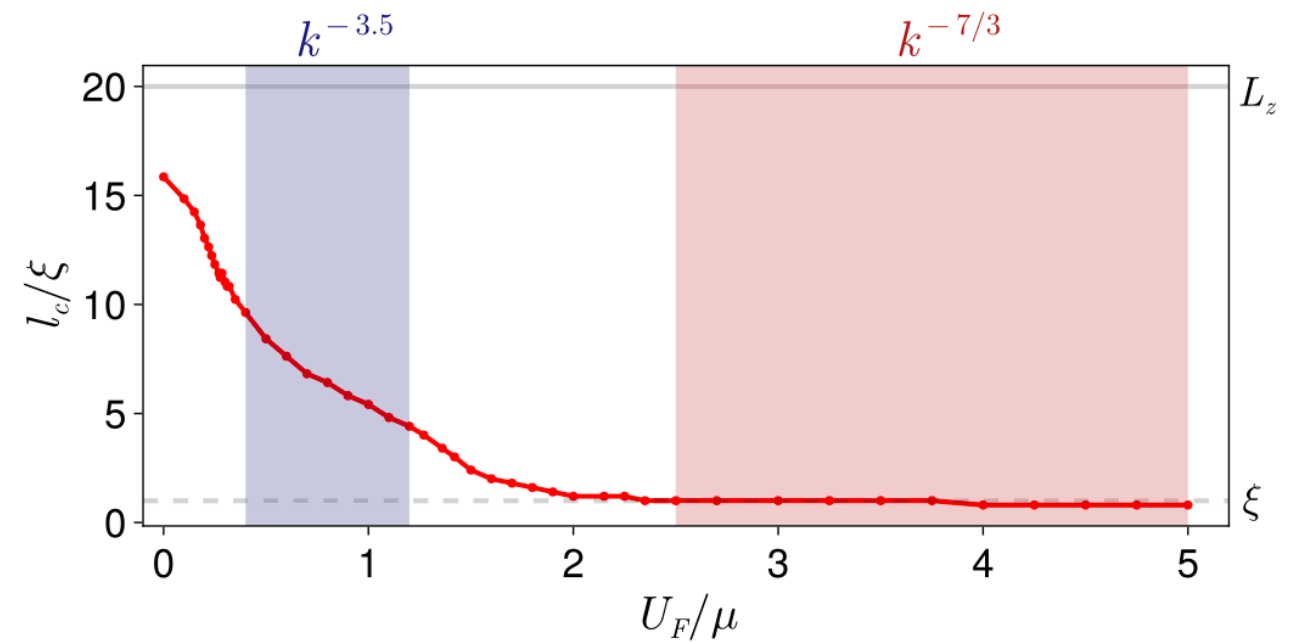
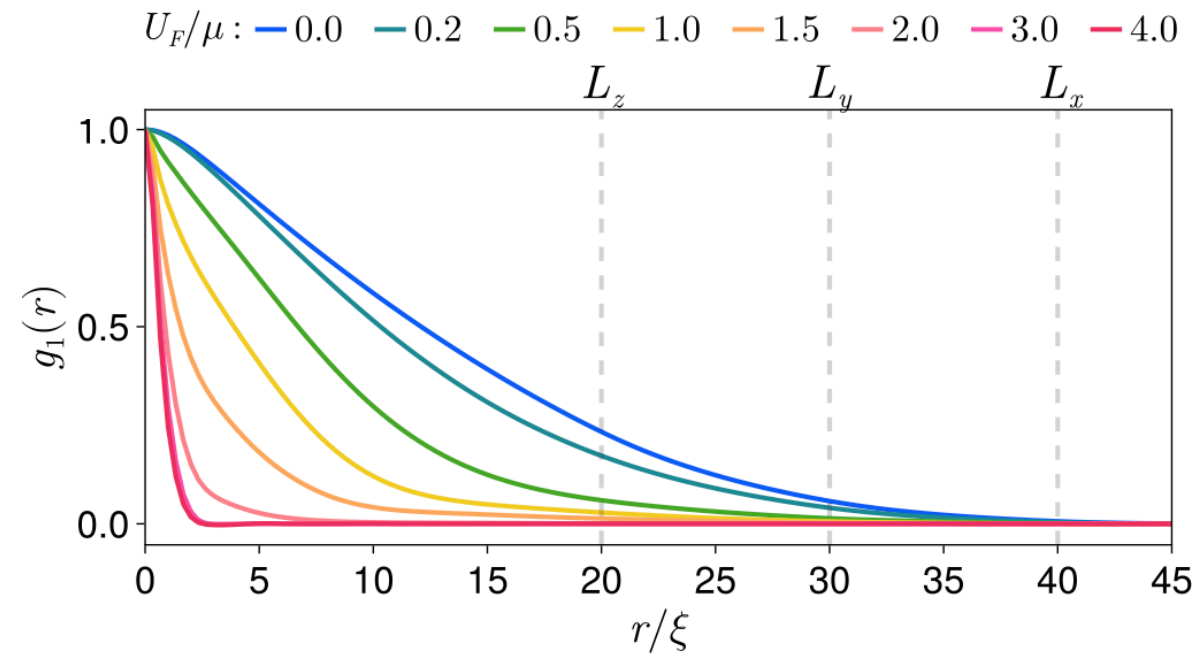
- Phonons on coherent background
- Quantum pressure = compressible
- Driven at scale $\lambda \sim 2\pi\xi$ (phonon-particle crossover scale)

4-wave, isotropic, **inverse particle cascade**

- Compressible inverse cascade
- Driven at healing length $\sim \xi$

Phase coherence collapses in the strong-vortex regime

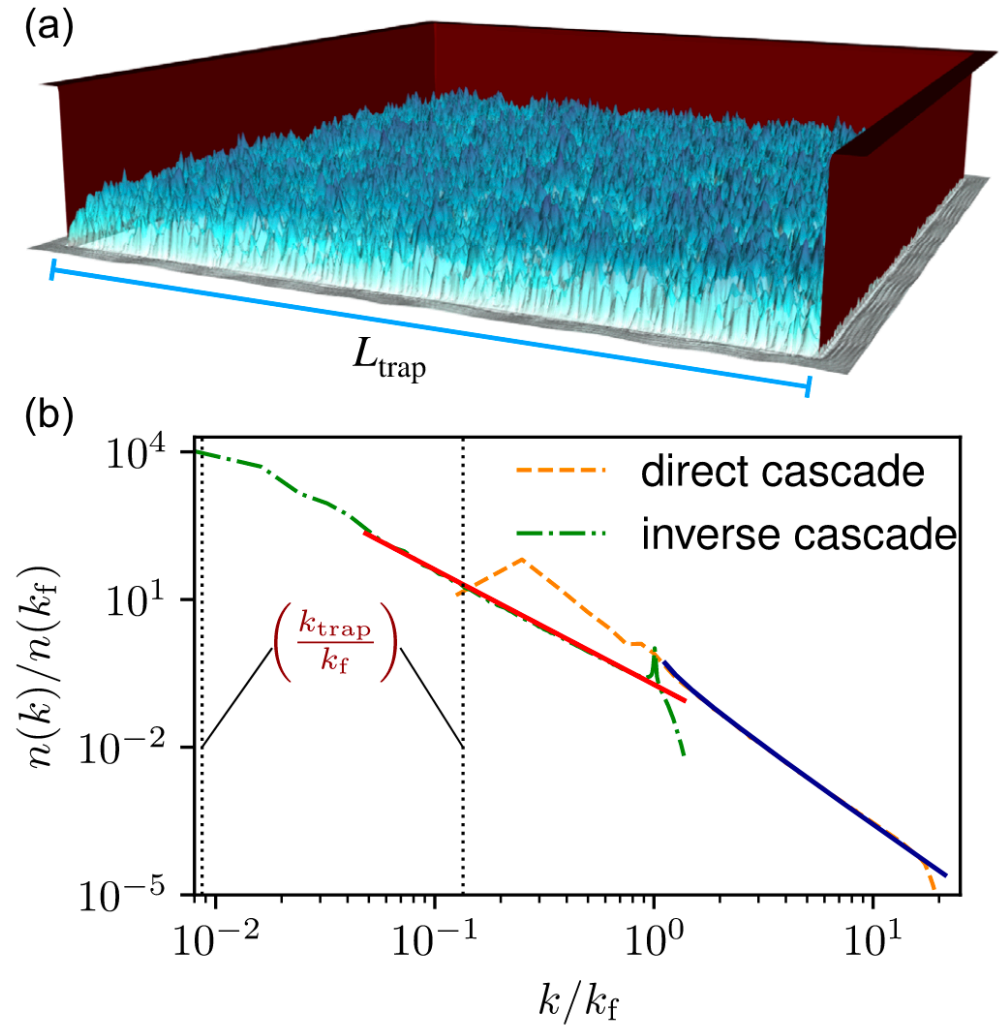
Stronger forcing shortens the coherence length, reducing to order ξ



Forced 3D turbulence

Analytic treatment of WWT

- Large scale simulations, analytical treatment of weak wave turbulence (WWT) regime
- Uses four-wave kinetic equation: “in the regime where the condensate is negligible”
- Log correction: $k^{-3.5}$ could be from non locality of interactions (three wave also gives k^{-3})

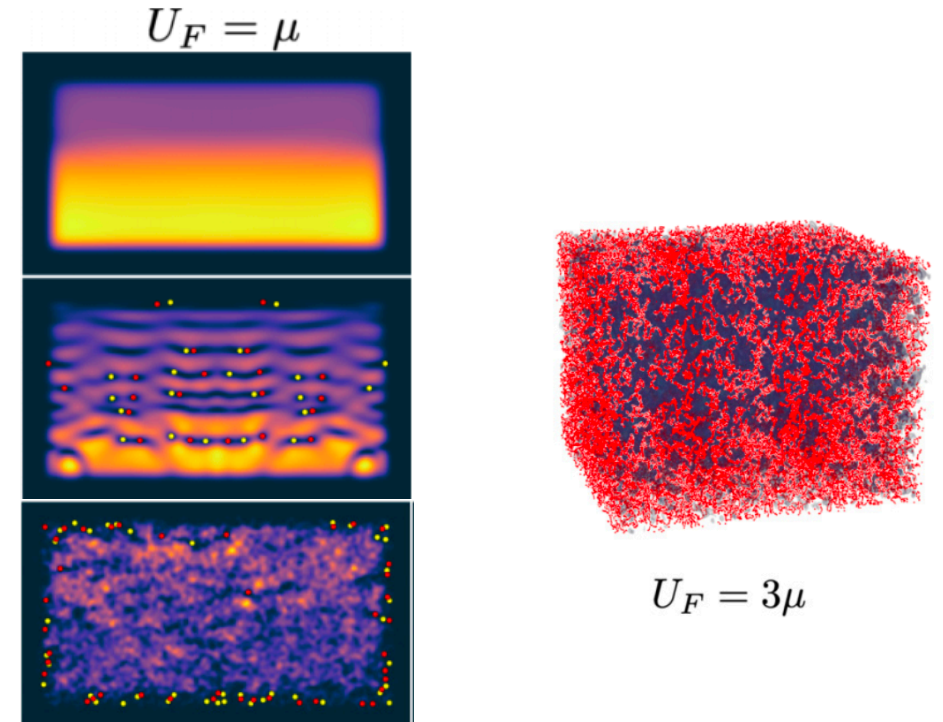
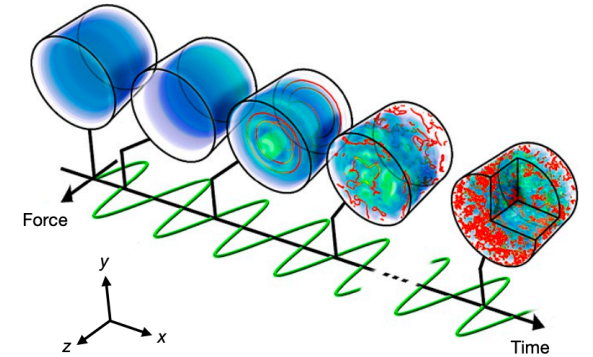


Summary, Perspectives

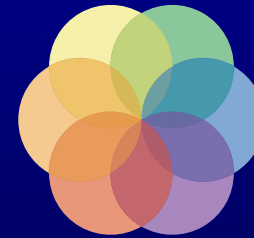
Library: QuantumFluidSpectra.jl

Steady state quantum fluid turbulence

1. Spectrum $n(k) \sim k^{-7/2}$ for weak waves as reported in the experiment [not ideal KZ WWT]. Driving scale identified as phonon-particle crossover scale.
2. Increasing forcing drives a transition to **bulk vortex turbulence**.
3. Vortex proliferation collapses phase coherence, observe **compressible** $k^{-7/3}$ **spectrum** [KZ WWT] drive at healing length scale, possibly by vortex annihilation.
4. Intermediate regime is also an **interesting quantum regime, remains an open problem**



Thank you!



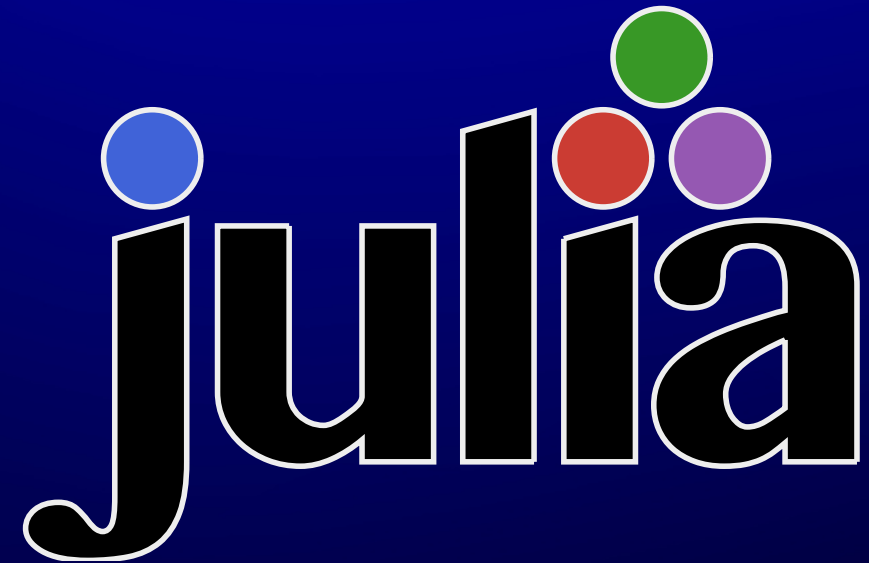
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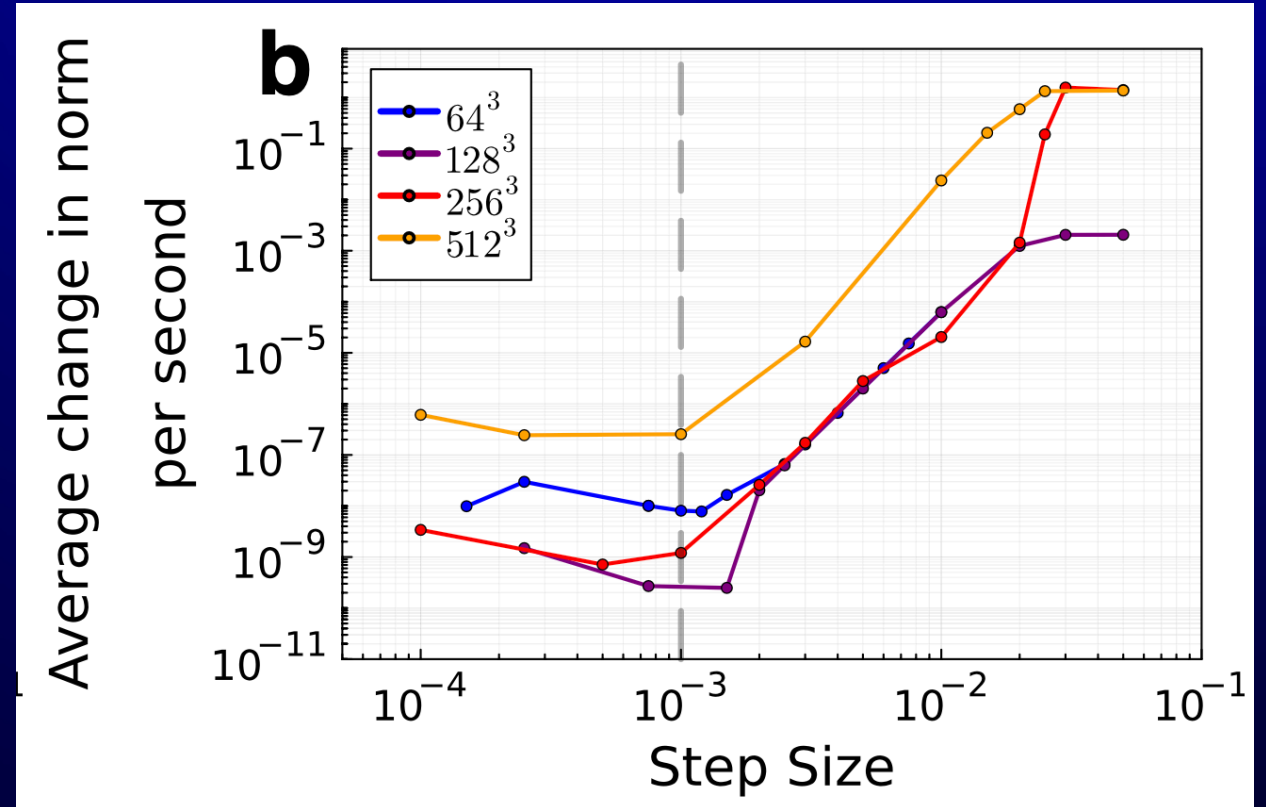
GPU simulations

- RK4IP: Fourth-order Runge-Kutta in the interaction picture (originated at Otago from Rob Ballagh group)
- Julia implementation at fixed step size, optimised for memory footprint; run on A100 GPUs
- Run time is about 38 minutes per physical second in Float64



Timestep

- Δt determined from highest energy simulation:
 $\Delta t = 7.6 \times 10^{-6} \text{s}$ (10^{-3} unitless)
- Finer grid degrades conservation due to floating point error
- Each simulation run for $t = 10\text{s} \sim 10^6$ time steps



Grid resolution

Convergence

- 256^3 and 512^3 are identical for physically relevant wavenumber at highest forcing amplitude
- We use 256^3 grid

